

COHOMOLOGY MOD 3 OF THE CLASSIFYING SPACE OF THE EXCEPTIONAL LIE GROUP E_6 , II : THE WEYL GROUP INVARIANTS

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ABSTRACT. We calculate the Weyl group invariants with respect to a maximal torus of the exceptional Lie group E_6 .

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1. INTRODUCTION.

The present paper is a sequel to [MST3].

Let E_6 be the compact, simply connected, exceptional Lie group of rank 6, T its maximal torus, BX the classifying space of X for $X = E_6$ or T , and $W(E_6)$ the Weyl group which acts on $H^*(BT; \mathbb{Z})$, and hence on $H^*(BT; \mathbb{Z}_3)$, as usual. As is well known, the invariant subalgebra $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ contains the image of the homomorphism $Bi^* : H^*(BE_6; \mathbb{Z}_3) \rightarrow H^*(BT; \mathbb{Z}_3)$.

The Rothenberg-Steenrod spectral sequence $\{E_r(X), d_r\}$ for X a compact associative H -space has

$$\begin{aligned} E_2(X) &= \text{Cotor}_A(\mathbb{Z}_p, \mathbb{Z}_p) \quad \text{with} \quad A = H^*(X; \mathbb{Z}_p), \\ E_\infty(X) &= \text{Gr} H^*(BX; \mathbb{Z}_p) \end{aligned}$$

where p is a prime number.

The E_2 -term $\text{Cotor}_A(\mathbb{Z}_p, \mathbb{Z}_p)$ for $X = E_6$ and $p = 3$ is calculated in [MS] (see also [MST3]). In this series of papers, it will be shown that the spectral sequence associated with E_6 collapses and the ring structure of $H^*(BE_6; \mathbb{Z}_3)$ will be determined (cf. [KM] where some of the ring structure of $H^*(BE_6; \mathbb{Z}_3)$ was determined).

The present paper is organized as follows. In Section 2, the action of the Weyl group $W(E_6)$ on $H^*(BT; \mathbb{Z}_3)$ will be described in terms of some generators of $H^*(BT; \mathbb{Z}_3)$. From this, 6 elements in $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ are immediately obtained, that is, $x_4, x_8, y_{10}, x_{20}, y_{22}$ and y_{26} . In Section 3, another set S of elements in $H^*(BT; \mathbb{Z}_3)$ which is invariant as a set under $W(E_6)$ is considered. The elementary symmetric functions on the elements of S are $W(E_6)$ -invariant and we find another element x_{36} . In Section 4, it is shown firstly that

$$H^*(BT; \mathbb{Z}_3)^{W(E_6)} \subset \mathbb{Z}_3(x_4)[x_8, y_{10}, x_{20}, y_{22}, x_{36}].$$

Then 6 more generators of $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ are totally determined explicitly. We prove

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Theorem 4.13 $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ is generated by the following thirteen elements:

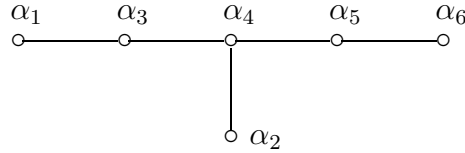
$$x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}, x_{36}, x_{48}, x_{54}, y_{58}, y_{60}, y_{64}, y_{76}.$$

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This paper has taken a long time to write. In fact, the results of the paper were announced in [T].

2. SOME INVARIANT ELEMENTS IN LOW DEGREES.

Let $T \subset E_6$ be a fixed maximal torus of E_6 , V_T the universal covering of T , V_T^* the dual of V_T . According to Bourbaki [B], the Dynkin diagram is given as follows:



where $\alpha_i \in V_T^*$ for $i = 1, 2, \dots, 6$ are the simple roots of E_6 . Let $\langle \cdot, \cdot \rangle$ be the invariant metric on the Lie algebra of E_6 , V_T and V_T^* , normalized in such a way that

$$\begin{aligned} \langle \alpha_i, \alpha_i \rangle &= 2, \\ \langle \alpha_i, \alpha_j \rangle &= -1 \quad \text{if } i \neq j \text{ and } \alpha_i \text{ and } \alpha_j \text{ are connected,} \\ \langle \alpha_i, \alpha_j \rangle &= 0 \quad \text{otherwise.} \end{aligned}$$

Let β_j be the corresponding fundamental weights: $\langle \beta_j, \alpha_i \rangle = \delta_{ij}$. Let us denote by R_i the reflection with respect to the hyperplane $\alpha_i = 0$. Then

$$R_i(\beta_i) = \beta_i - \sum_j \langle \alpha_i, \alpha_j \rangle \beta_j \quad \text{and} \quad R_i(\beta_j) = \beta_j \quad \text{for } i \neq j.$$

Denote by U the centralizer of the torus T^1 defined by $\langle \alpha_i, t \rangle = 0$ for $i = 2, 3, \dots, 6$ and $t \in T$. Then U is a closed connected subgroup of maximal rank and of local type $D_5 \times T^1$ such that $D_5 \cap T^1 = \mathbb{Z}_4$ (see [V]). The Weyl groups $W(E_6)$ and $W(U)$ are generated by R_1, R_2, \dots, R_6 and R_2, \dots, R_6 , respectively.

Put

$$\begin{aligned} \tau_6 &= \beta_6, \\ \tau_5 &= R_6(\tau_6) = \beta_5 - \beta_6, \\ \tau_4 &= R_5(\tau_5) = \beta_4 - \beta_5, \\ \tau_3 &= R_4(\tau_4) = \beta_2 + \beta_3 - \beta_4, \\ \tau_2 &= R_3(\tau_3) = \beta_1 + \beta_2 - \beta_3, \\ \tau_1 &= R_1(\tau_2) = -\beta_1 + \beta_2, \\ x &= \beta_2 = \frac{1}{3}(\tau_1 + \tau_2 + \dots + \tau_6). \end{aligned} \tag{2.1}$$

Then β_i are linear combinations of τ_j 's and x as follows:

$$\begin{aligned}\beta_1 &= x - \tau_1, & \beta_2 &= x, & \beta_3 &= -x + \tau_3 + \tau_4 + \tau_5 + \tau_6, \\ \beta_4 &= \tau_4 + \tau_5 + \tau_6, & \beta_5 &= \tau_5 + \tau_6, & \beta_6 &= \tau_6.\end{aligned}$$

Further we put

$$t = x - \tau_1 \quad \text{and} \quad t_i = \tau_{i+1} - \frac{1}{2}t \quad \text{for} \quad i = 1, \dots, 5.$$

Then we have

$$(2.2) \quad \begin{aligned}t &= \beta_1, \\ t_1 &= \frac{1}{2}\beta_1 + \beta_2 - \beta_3, \\ t_2 &= -\frac{1}{2}\beta_1 + \beta_2 + \beta_3 - \beta_4, \\ t_3 &= -\frac{1}{2}\beta_1 + \beta_4 - \beta_5, \\ t_4 &= -\frac{1}{2}\beta_1 + \beta_5 - \beta_6, \\ t_5 &= -\frac{1}{2}\beta_1 + \beta_6,\end{aligned}$$

and

$$(2.2)' \quad \begin{aligned}\beta_1 &= t, \\ \beta_2 &= \frac{3}{4}t + \frac{1}{2}t_1 + \frac{1}{2}t_2 + \frac{1}{2}t_3 + \frac{1}{2}t_4 + \frac{1}{2}t_5, \\ \beta_3 &= \frac{5}{4}t - \frac{1}{2}t_1 + \frac{1}{2}t_2 + \frac{1}{2}t_3 + \frac{1}{2}t_4 + \frac{1}{2}t_5, \\ \beta_4 &= \frac{3}{2}t + t_3 + t_4 + t_5, \\ \beta_5 &= t + t_4 + t_5, \\ \beta_6 &= \frac{1}{2}t + t_5.\end{aligned}$$

Denote $t_1 + t_2 + t_3 + t_4 + t_5$ by c_1 . Then we have

$$(2.3) \quad c_1 = t_1 + t_2 + \dots + t_5 = \tau_2 + \tau_3 + \dots + \tau_6 - \frac{5}{2}t = 2x - \frac{3}{2}t.$$

The R_i -operations are given by

| | R_1 | R_2 | R_3 | R_4 | R_5 | R_6 |
|-------|--|--------|-------|-------|-------|-------|
| t | $\frac{1}{4}t - t_1 + \frac{1}{2}c_1$ | | | | | |
| t_1 | $-\frac{3}{8}t + \frac{1}{2}t_1 + \frac{1}{4}c_1$ | $-t_2$ | t_2 | | | |
| t_2 | $\frac{3}{8}t + \frac{1}{2}t_1 + t_2 - \frac{1}{4}c_1$ | $-t_1$ | t_1 | t_3 | | |
| t_3 | $\frac{3}{8}t + \frac{1}{2}t_1 + t_3 - \frac{1}{4}c_1$ | | | t_2 | t_4 | |
| t_4 | $\frac{3}{8}t + \frac{1}{2}t_1 + t_4 - \frac{1}{4}c_1$ | | | | t_3 | t_5 |
| t_5 | $\frac{3}{8}t + \frac{1}{2}t_1 + t_5 - \frac{1}{4}c_1$ | | | | | t_4 |

where the blanks indicate the trivial action. Taking coefficients in \mathbb{Z}_3 , we have

$$(2.4) \quad \begin{aligned}R_1(t) &= t - (t_1 + c_1), \\ R_1(t_1) &= -t_1 + c_1, \\ R_1(t_i) &= t_i - (t_1 + c_1) \quad \text{for } i = 2, 3, 4, 5.\end{aligned}$$

Roots or weights will be considered in the usual way (cf. [BH]) as elements of $H^*(T)$, $H^*(T; \mathbb{Z}_3)$, $H^*(BT)$ or $H^*(BT; \mathbb{Z}_3)$. Then the weights β_1, \dots, β_6 generate $H^*(BT)$ as well

as $H^*(BT; \mathbb{Z}_3)$ and so do the elements t, t_1, \dots, t_5 :

$$H^*(BT; \mathbb{Z}_3) = \mathbb{Z}_3[t, t_1, \dots, t_5] \quad \text{with} \quad t, t_1, t_2, \dots, t_5 \in H^2(BT; \mathbb{Z}_3).$$

From now on until the end of Section 2 all coefficients are in \mathbb{Z}_3 .

The Weyl group $W(E_6)$ acts on BT and hence on $H^*(BT; \mathbb{Z}_3)$, and the invariant subalgebra $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ contains the image of the homomorphism $Bi^* : H^*(BE_6; \mathbb{Z}_3) \rightarrow H^*(BT; \mathbb{Z}_3)$ induced by the natural map $Bi : BT \rightarrow BE_6$. The action of $W(U)$ is the same as the usual action of $W(SO(10))$. Thus we obtain

$$(2.5) \quad \begin{aligned} \text{Im } Bi^*(H^*(BE_6; \mathbb{Z}_3)) &\subset H^*(BT; \mathbb{Z}_3)^{W(E_6)} \\ &\subset H^*(BT; \mathbb{Z}_3)^{W(U)} = \mathbb{Z}_3[t, p_1, p_2, c_5, p_3, p_4] \end{aligned}$$

where $c_i = \sigma_i(t_1, t_2, \dots, t_5)$ and $p_i = \sigma_i(t_1^2, t_2^2, \dots, t_5^2)$ are the elementary symmetric functions on t_i and t_j^2 , respectively.

We shall determine $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$, that is, R_1 -invariant elements in $\mathbb{Z}_3[t, p_1, p_2, c_5, p_3, p_4]$.

We obtain, from the simple relation between c_i 's and p_j 's, that

$$(2.6) \quad \begin{aligned} c_2 &= p_1 - c_1^2, \\ c_4 &= -p_2 + c_1^4 + c_1^2 p_1 + c_1 c_3 + p_1^2, \\ c_3 c_5 &= p_4 - c_4^2, \\ c_3^2 &= p_3 + c_1^6 + c_1^3 c_3 - c_1^2 p_2 - c_1 c_3 p_1 + c_1 c_5 - p_1^3 + p_1 p_2. \end{aligned}$$

The equality $0 = \prod_{i=1}^5 (t_1 - t_i)$ gives rise to

$$t_1^5 = c_5 - t_1 c_4 + t_1^2 c_3 - t_1^3 c_2 + t_1^4 c_1.$$

Put $b = t_1 + c_1$, then we have

$$(2.7) \quad \begin{aligned} b^5 &= (-c_1^5 - c_1^3 p_1 - c_1^2 c_3 + c_1 p_1^2 - c_1 p_2 + c_5) \\ &\quad + b(-c_1^4 - c_1^2 p_1 - p_1^2 + p_2) + b^2(c_1^3 + c_3) - b^3(c_1^2 + p_1). \end{aligned}$$

From (2.4), and replacing $c_2, c_4, c_3 c_5, c_3^2$ and b^5 by (2.6) and (2.7), we have

$$(2.8) \quad \begin{aligned} R_1(t) &= t - b, \\ R_1(b) &= -b, \\ R_1(c_1) &= c_1, \\ R_1(p_1) &= p_1, \\ R_1(p_2) &= p_2, \\ R_1(c_3) &= c_3 - b(c_1^2 + p_1) + b^3, \\ R_1(c_5) &= c_5 + b(p_1^2 - p_2) - b^3 p_1, \\ R_1(p_3) &= p_3 + b(-c_1^3 p_1 - c_1 p_1^2 + c_3 p_1 + c_5) + b^2(c_1^2 p_1 + p_2) - b^3 c_1 p_1, \\ R_1(p_4) &= p_4 + b(-c_1^5 p_1 + c_1^3 p_1^2 + c_1^3 p_2 - c_1^2 c_3 p_1 + c_3 p_1^2 - c_3 p_2 + c_5 p_1) \\ &\quad + b^2(-c_1^4 p_1 - c_1^2 p_2) + b^3(c_1^3 p_1 - c_1 p_1^2 + c_1 p_2 + c_3 p_1 + c_5) \\ &\quad + b^4(-c_1^2 p_1 - p_2). \end{aligned}$$

Thus the following three elements are clearly R_1 -invariant and hence $W(E_6)$ -invariant:

$$(2.9) \quad x_4 = p_1, \quad x_8 = p_2 - p_1^2, \quad y_{10} = c_5 - t x_8 - t^3 x_4.$$

Put

$$(2.10) \quad h_{12} = p_3 + ty_{10} - t^2x_8 - t^4x_4 \quad \text{and} \quad h_{16} = p_4 + t^3y_{10} - t^4x_8 - t^6x_4.$$

Then we see that

$$(2.11) \quad R_1(h_{12}) = h_{12} + d_8x_4 \quad \text{and} \quad R_1(h_{16}) = h_{16} - d_8x_8,$$

where

$$(2.12) \quad d_8 = b(-t^3 - c_1^3 - c_1x_4 + c_3) + b^2(c_1^2 + x_4) + b^3(t - c_1) - b^4$$

and

$$(2.13) \quad R_1(d_8) = -d_8.$$

It follows that

$$\mathbb{Z}_3[t, p_1, p_2, c_5, p_3, p_4] = \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}].$$

Further, we put

$$(2.14) \quad h_{18} = t(h_{16} - x_8^2) + t^3(-h_{12} + x_4x_8) + t^5x_8 - t^7x_4 + t^9$$

which is taken so as to satisfy

$$(2.15) \quad R_1(h_{18}) = h_{18} + d_8y_{10}.$$

It follows from (2.11) and (2.15) that the following three elements are R_1 -invariant and hence $W(E_6)$ -invariant:

$$(2.16) \quad \begin{aligned} x_{20} &= h_{12}x_8 + h_{16}x_4, \\ y_{22} &= h_{12}y_{10} - h_{18}x_4, \\ y_{26} &= h_{16}y_{10} + h_{18}x_8, \end{aligned}$$

among which holds the following relation:

$$(2.17) \quad -x_{20}y_{10} + y_{22}x_8 + y_{26}x_4 = 0.$$

Summing up we have found the first six $W(G)$ -invariant elements:

$$x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}.$$

3. THE ELEMENT x_{36} .

In order to obtain another R_1 -invariant element, we consider the following set S (cf. [TW]). Put

$$(3.1) \quad w_i = 2\tau_i - x \quad \text{for} \quad i = 1, 2, \dots, 6$$

and denote by S the set

$$\{ w_i + w_j \text{ for } i < j, \quad x - w_i, \quad -x - w_i \}.$$

We see that the set S is invariant as a set under the action of $W(E_6)$. Therefore the elementary symmetric functions σ_i^S on the 27 elements of S are invariant under the action of $W(E_6)$.

We shall calculate

$$(3.2) \quad \begin{aligned} P &= 1 + \sum_{j=1}^{27} \sigma_j^S = \prod_{y \in S} (1 + y) \\ &= \prod_{1 \leq i < j \leq 6} (1 + w_i + w_j) \cdot \prod_{1 \leq j \leq 6} (1 + x - w_j) \cdot \prod_{1 \leq j \leq 6} (1 - x - w_j) \end{aligned}$$

and decompose the result by degree to obtain σ_j^S .

Since we have

$$x = -c_1, \quad w_1 = t - c_1 \quad \text{and} \quad w_i = t + c_1 - t_{i-1} \quad \text{for } i > 1$$

by (3.1), (2.1), (2.2)' and (2.3), the polynomial P is expressed in terms of c_i 's and t . (The calculation is carried out by Mathematica and the result has 2600 terms.)

We have from (2.6), (2.9) and (2.10) :

$$(3.3) \quad \begin{aligned} c_2 &= -c_1^2 + x_4, \\ c_4 &= c_1^4 + c_1 c_3 + c_1^2 x_4 - x_8, \\ c_5 &= y_{10} + x_8 t + x_4 t^3, \\ c_3^2 &= h_{12} + t^4 x_4 + t^2 x_8 - t y_{10} + x_4 x_8 + t^3 x_4 c_1 + t x_8 c_1 - x_4^2 c_1^2 - x_4 c_1 c_3 - x_8 c_1^2 \\ &\quad + y_{10} c_1 + c_1^6 + c_1^3 c_3, \\ c_3 y_{10} &= h_{16} + t^6 x_4 + t^4 x_8 - t^3 y_{10} - x_8^2 - t^4 x_4 c_1^2 - t^3 x_4 c_1^3 - t^3 x_4 c_3 - t^2 x_8 c_1^2 - t x_8 c_1^3 \\ &\quad - t x_8 c_3 + t y_{10} c_1^2 - h_{12} c_1^2 + x_4 x_8 c_1^2 + x_4 c_1^6 - x_4 c_1^3 c_3 - x_8 c_1 c_3 - y_{10} c_1^3 + c_1^8. \end{aligned}$$

Rewrite P making use of (3.3) and we find no c_3 (as well as c_2 , c_4 and c_5) in the result P_1 and $P_1 \in \mathbb{Z}_3[c_1, t, x_4, x_8, y_{10}, h_{12}, h_{16}]$.

We use two more relations to get rid of c_1 's.

Replacing c_3^2 in the equality $c_3^2 y_{10} - c_3 \cdot c_3 y_{10} = 0$, we obtain a relation:

$$(3.4) \quad \begin{aligned} h_{16} c_3 &= t^9 x_4 + t^7 x_4^2 + t^7 x_8 - t^6 y_{10} - t^5 x_4 x_8 + t^3 h_{12} x_4 + t^3 h_{16} + t^3 x_4^2 x_8 \\ &\quad + t h_{12} x_8 + t x_4 x_8^2 - t y_{10}^2 + h_{12} y_{10} + x_4 x_8 y_{10} + t^7 x_4 c_1^2 + t^6 x_4 c_1^3 + t^6 x_4 c_3 \\ &\quad + t^5 x_4 c_1^4 + t^5 x_8 c_1^2 - t^4 x_4^2 c_1^3 - t^4 x_4 x_8 c_1 - t^4 x_4 c_1^5 - t^4 x_4 c_1^2 c_3 + t^4 x_8 c_1^3 \\ &\quad + t^4 x_8 c_3 - t^4 y_{10} c_1^2 - t^3 h_{12} c_1^2 - t^3 x_4^2 c_1^2 - t^3 x_4^2 c_1^4 + t^3 x_4 x_8 c_1^2 - t^3 x_4 c_1^3 c_3 \\ &\quad + t^3 x_8 c_1^4 - t^3 x_8 c_1 c_3 + t^3 c_1^8 - t^2 x_4 x_8 c_1^3 - t^2 x_8^2 c_1 - t^2 x_8 c_1^5 - t^2 x_8 c_1^2 c_3 \\ &\quad - t^2 y_{10} c_1^4 + t h_{12} c_1^4 - t h_{16} c_1^2 - t x_4^2 x_8 c_1^2 + t x_4 x_8 c_1^4 + t x_4 y_{10} c_1^3 - t x_4 c_1^8 \\ &\quad + t x_4 c_1^5 c_3 + t x_8^2 c_1^2 + t x_8 y_{10} c_1 - t x_8 c_1^6 + t x_8 c_1^3 c_3 - t c_1^{10} - h_{12} x_4 c_1^3 \\ &\quad + h_{12} x_8 c_1 + h_{12} c_1^5 + h_{12} c_1^2 c_3 - h_{16} x_4 c_1 - h_{16} c_1^3 - x_4^2 c_1^5 - x_4^2 x_8 c_1^3 \\ &\quad - x_4^2 y_{10} c_1^2 - x_4^2 c_1^7 - x_4 x_8^2 c_1 + x_4 x_8 c_1^5 - x_4 x_8 c_1^2 c_3 - x_4 y_{10} c_1^4 - x_4 c_1^9 \\ &\quad + x_4 c_1^6 c_3 + x_8^2 c_3 + x_8 c_1^7 - x_8 c_1^4 c_3 + y_{10}^2 c_1 - y_{10} c_1^6 - c_1^{11} - c_1^8 c_3, \end{aligned}$$

Rewriting c_3^2 , c_3y_{10} and c_3h_{16} in the equality $c_3^2y_{10}^2 - (c_3y_{10})^2 = 0$ gives rise to the following relation.

$$\begin{aligned}
(3.5) \quad c_1^{16} = & -t^{12}x_4^2 + t^{10}x_4^3 + t^{10}x_4x_8 - t^9x_4y_{10} - t^8x_8^2 + t^7x_4^2y_{10} - t^7x_8y_{10} \\
& + t^6h_{12}x_4^2 + t^6h_{16}x_4 + t^6x_4^3x_8 - t^6x_4x_8^2 - t^6y_{10}^2 - t^5x_4x_8y_{10} \\
& - t^4h_{12}x_4x_8 + t^4h_{16}x_8 - t^4x_4^2x_8^2 - t^4x_4y_{10}^2 - t^3h_{12}x_4y_{10} - t^3h_{16}y_{10} \\
& - t^3x_4^2x_8y_{10} - t^3x_8^2y_{10} + t^2h_{12}x_8^2 + t^2x_4x_8^3 - t^2x_8y_{10}^2 - th_{12}x_8y_{10} \\
& - tx_4x_8^2y_{10} - ty_{10}^3 + h_{12}y_{10}^2 - h_{16}^2 - h_{16}x_8^2 + x_4x_8y_{10}^2 - x_8^4 - t^{10}x_4^2c_1^2 \\
& - t^8x_4^2c_1^4 + t^8x_4x_8c_1^2 - t^7x_4y_{10}c_1^2 - t^6h_{12}x_4c_1^2 - t^6x_4^4c_1^2 - t^6x_4^3c_1^4 \\
& + t^6x_4^2x_8c_1^2 + t^6x_4^2c_1^6 - t^6x_4x_8c_1^4 + t^6x_4c_1^8 - t^6x_8^2c_1^2 - t^5x_4y_{10}c_1^4 \\
& - t^5x_8y_{10}c_1^2 + t^4h_{12}x_4c_1^4 - t^4h_{12}x_8c_1^2 - t^4h_{16}x_4c_1^2 + t^4x_4^3x_8c_1^2 - t^4x_4^3c_1^6 \\
& + t^4x_4^2x_8c_1^4 + t^4x_4^2c_1^8 + t^4x_4x_8^2c_1^2 - t^4x_4x_8c_1^6 - t^4x_4c_1^{10} + t^4x_8c_1^8 - t^4y_{10}^2c_1^2 \\
& - t^3h_{12}x_4x_8c_1 + t^3h_{12}y_{10}c_1^2 - t^3h_{16}x_4^2c_1 + t^3x_4^4c_1^5 - t^3x_4^3x_8c_1^3 + t^3x_4^3y_{10}c_1^2 \\
& + t^3x_4^3c_1 + t^3x_4^2x_8c_1^5 + t^3x_4^2y_{10}c_1^4 - t^3x_4x_8^2c_1^3 - t^3x_4x_8y_{10}c_1^2 + t^3x_4x_8c_1^7 \\
& + t^3x_4y_{10}^2c_1 - t^3x_4y_{10}c_1^6 + t^3x_8y_{10}c_1^4 - t^3y_{10}c_1^8 + t^2h_{12}x_8c_1^4 - t^2h_{16}x_8c_1^2 \\
& - t^2x_4^2x_8^2c_1^2 - t^2x_4^2x_8c_1^6 - t^2x_4x_8^2c_1^4 + t^2x_4x_8c_1^8 + t^2x_8^2c_1^6 - t^2x_8c_1^{10} \\
& - t^2y_{10}^2c_1^4 - th_{12}x_8^2c_1 - th_{12}y_{10}c_1^4 - th_{16}x_4x_8c_1 + th_{16}y_{10}c_1^2 + tx_4^3x_8c_1^5 \\
& - tx_4^2x_8^2c_1^3 + tx_4^2x_8y_{10}c_1^2 + tx_4^2x_8c_1^7 + tx_4^2y_{10}c_1^6 + tx_4x_8^2c_1^5 + tx_4x_8y_{10}c_1^4 \\
& - tx_4y_{10}c_1^8 - tx_8^3c_1 + tx_8^2c_1^7 + tx_8y_{10}^2c_1 - tx_8y_{10}c_1^6 + ty_{10}c_1^{10} - h_{12}^2c_1^4 \\
& - h_{12}h_{16}c_1^2 - h_{12}x_4^2c_1^6 - h_{12}x_4x_8c_1^4 + h_{12}x_4c_1^8 - h_{12}x_8^2c_1^2 - h_{12}x_8y_{10}c_1 \\
& - h_{12}x_8c_1^6 - h_{12}c_1^{10} - h_{16}x_4^2c_1^4 - h_{16}x_4y_{10}c_1 - h_{16}x_4c_1^6 + h_{16}x_8c_1^4 + h_{16}c_1^8 \\
& - x_4^4c_1^8 + x_4^3x_8c_1^6 + x_4^3y_{10}c_1^5 - x_4^3c_1^{10} - x_4^2x_8y_{10}c_1^3 - x_4^2y_{10}^2c_1^2 + x_4^2y_{10}c_1^7 \\
& + x_4x_8^3c_1^2 + x_4x_8y_{10}c_1^5 - x_4c_1^{14} + x_8^3c_1^4 - x_8^2y_{10}c_1^3 + x_8y_{10}^2c_1^2 \\
& + x_8y_{10}c_1^7 + x_8c_1^{12} + y_{10}^3c_1 - y_{10}^2c_1^6.
\end{aligned}$$

Rewrite c_1^{16} in P_1 and the result P_2 has no c_1 and $P_2 \in \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]$.

The highest degree of t is 27. We use the relations obtained from (2.14), (2.16) and (2.17) to lower the degree of t to 8 :

$$(3.6) \quad t^9 = h_{18} + t^7x_4 - t^5x_8 + t^3h_{12} - t^3x_4x_8 - th_{16} + tx_8^2,$$

$$(3.7) \quad h_{18}x_8 = -h_{16}y_{10} + y_{26},$$

$$h_{18}x_4 = h_{12}y_{10} - y_{22},$$

$$h_{16}x_4 = -h_{12}x_8 + x_{20},$$

$$y_{26}x_4 = x_{20}y_{10} - y_{22}x_8,$$

$$(3.8) \quad h_{18}x_{20} = h_{12}y_{26} - h_{16}y_{22}.$$

The result P_3 is as follows.

(3.9)

$$\begin{aligned}
P_3 = & 1 + h_{12}x_4y_{26} - h_{12}x_4^2x_8^2 + h_{12}x_4^2y_{10}t^3 + h_{12}x_8y_{22} + h_{12}x_{20}x_4 - h_{12}x_{20}y_{10} + h_{12}^2x_4x_8 \\
& - h_{12}^3 - h_{16}h_{18}y_{10}^2 + h_{16}x_8y_{10}^3 - h_{16}x_8^2y_{10}t^3 + h_{16}x_8^4 + h_{16}^2x_8^2 + h_{16}^3 - h_{18}y_{10}y_{26} \\
& + h_{18}y_{10}^3t^3 - h_{18}^3 - x_4x_8 + x_4x_8y_{10}^3 + x_4x_8y_{10}^3t^6 + x_4x_8y_{10}^4 - x_4x_8y_{10}^4t + x_4x_8^2y_{10} \\
& - x_4x_8^2y_{10}^2 + x_4x_8^2y_{26}t + x_4x_8^3y_{10}^2 + x_4x_8^4 + x_4x_8^4t^6 - x_4x_8^4y_{10}t - x_4y_{10}y_{22} + x_4y_{10}^2 \\
& + x_4y_{10}^3 - x_4y_{26} - x_4^2x_8 + x_4^2x_8y_{10}^2 - x_4^2x_8y_{10}^3 + x_4^2x_8y_{10}^3t^4 + x_4^2x_8y_{26} - x_4^2x_8^2y_{10} \\
& + x_4^2x_8^3y_{10}t^3 + x_4^2x_8^4 + x_4^2x_8^4t^4 - x_4^2y_{10} - x_4^2y_{10}^2 - x_4^2y_{10}^4 + x_4^2y_{10}^4t^3 + x_4^2y_{22} - x_4^2y_{22}t^3 \\
& + x_4^2y_{26} - x_4^3 - x_4^3x_8t^8 - x_4^3x_8y_{10}t^3 - x_4^3x_8^2 - x_4^3x_8^2t^4 - x_4^3x_8^3 - x_4^3x_8^3t^6 + x_4^3y_{10}t^7 \\
& + x_4^3y_{10}^2 - x_4^3y_{10}^2t^2 + x_4^3y_{10}^3 - x_4^3y_{10}^3t^6 + x_4^3y_{22} - x_4^3y_{22}t - x_4^4x_8 + x_4^4x_8t^6 + x_4^4x_8y_{10} \\
& - x_4^4x_8y_{10}t - x_4^5x_8 + x_4^5x_8t^4 - x_4^5y_{10} + x_4^5y_{10}t^3 + x_4^6 - x_4^6t^6 - x_8y_{10} + x_8y_{10}^2 \\
& - x_8y_{10}^2y_{26} - x_8y_{10}^3t^8 + x_8y_{10}^4 - x_8y_{10}^4t^3 - x_8y_{26} - x_8^2 - x_8^2y_{10}y_{26} - x_8^2y_{10}^2 - x_8^2y_{10}^3 \\
& - x_8^2y_{10}^3t^4 + x_8^2y_{26} + x_8^2y_{26}t^3 - x_8^3 + x_8^3y_{10} + x_8^3y_{10}t^7 + x_8^3y_{10}^2 - x_8^3y_{10}^2t^2 - x_8^4t^8 \\
& - x_8^4y_{10}t^3 - x_8^4y_{10}^2 + x_8^5 - x_8^5t^4 - x_{20}x_4 + x_{20}x_4x_8^2 + x_{20}x_4^2 - x_{20}x_4^2x_8 - x_{20}x_4^3 \\
& + x_{20}x_4^3t^2 - x_{20}x_8 - x_{20}x_8y_{10}^2 - x_{20}x_8^2 - x_{20}x_8^2y_{10}t - x_{20}x_8^3 + x_{20}x_8^3t^2 - x_{20}y_{10} \\
& - x_{20}y_{10}^2 - x_{20}y_{10}^3 + x_{20}y_{10}^3t^2 - x_{20}y_{22} + x_{20}y_{26} - x_{20}^2 - y_{10}y_{26} + y_{10}^2y_{22} - y_{10}^2y_{26} \\
& - y_{10}^3 + y_{10}^3y_{22} - y_{10}^3y_{22}t - y_{10}^4 + y_{10}^4t^7 + y_{10}^5 - y_{10}^5t^2 + y_{22}y_{26} - y_{26}^2
\end{aligned}$$

Now we decompose P_3 by degree to obtain σ_j^S .

The σ_j^S for $j \leq 17$ are as follows:

$$\begin{aligned}
(3.10) \quad & \sigma_i^S = 0 \quad \text{for } 1 \leq i \leq 5 \text{ and } i = 7, 10, 11, 13, \\
& \sigma_6^S = -x_4^3 - x_4x_8, \\
& \sigma_8^S = -x_4^2x_8 - x_8^2, \\
& \sigma_9^S = -x_4^2y_{10} - x_8y_{10}, \\
& \sigma_{12}^S = -x_{20}x_4 + x_4^6 - x_4^4x_8 + x_4y_{10}^2 - x_8^3, \\
& \sigma_{14}^S = x_{20}(x_4^2 - x_8) - x_4^5x_8 - x_4^3x_8^2 - x_4^2y_{10}^2 + x_8y_{10}^2, \\
& \sigma_{15}^S = x_{20}y_{10} + y_{22}(x_4^2 + x_8) - x_4^5y_{10} + x_4x_8^2y_{10} - y_{10}^3, \\
& \sigma_{16}^S = -x_{20}x_4^3 + x_4^3y_{10}^2, \\
& \sigma_{17}^S = x_{20}x_4y_{10} + y_{22}(x_4^3 - x_4x_8) - y_{26}x_8 \\
& \quad + x_4^4x_8y_{10} - x_4^2x_8^2y_{10} + x_4y_{10}^3 + x_8^3y_{10}.
\end{aligned}$$

The element σ_{18}^S is expressed as

$$\begin{aligned}
\sigma_{18}^S = & -x_{20}x_4^2x_8 - x_{20}x_8^2 - y_{22}x_4y_{10} - y_{26}y_{10} - x_4^3x_8^3 + x_4^2x_8y_{10}^2 + x_4x_8^4 \\
& - x_8^2y_{10}^2 - h_{12}^3 + h_{12}^2x_4x_8 + h_{12}x_{20}x_4 - h_{12}x_4^2x_8^2 - t^8x_4^3x_8 + t^7x_4^3y_{10} \\
& - t^6x_4^6 + t^6x_4^4x_8 + t^4x_4^5x_8 - t^4x_4^3x_8^2 + t^3h_{12}x_4^2y_{10} - t^3y_{22}x_4^2 \\
& + t^3x_4^5y_{10} - t^3x_4^3x_8y_{10} + t^2x_{20}x_4^3 - t^2x_4^3y_{10}^2 - ty_{22}x_4^3 - tx_4^4x_8y_{10}
\end{aligned}$$

which cannot be expressed in terms of the R_1 -invariant elements already obtained. We put

$$(3.11) \quad \begin{aligned} g_{24} = & -t^8x_8 + t^7y_{10} - t^6x_4^3 + t^6x_4x_8 + t^4x_4^2x_8 - t^4x_8^2 + t^3h_{18} \\ & + t^3x_4^2y_{10} - t^3x_8y_{10} + t^2x_{20} - t^2y_{10}^2 - ty_{22} - tx_4x_8y_{10} \end{aligned}$$

and

$$(3.12) \quad x_{36} = -g_{24}x_4^3 + h_{12}^3 - h_{12}^2x_4x_8 - h_{12}x_{20}x_4 + h_{12}x_4^2x_8^2.$$

Then

$$\begin{aligned} R_1(h_{12}^3 - h_{12}^2x_4x_8 - h_{12}x_{20}x_4 + h_{12}x_4^2x_8^2) \\ = (h_{12}^3 - h_{12}^2x_4x_8 - h_{12}x_{20}x_4 + h_{12}x_4^2x_8^2) \\ + x_4^3(d_8^3 - d_8^2x_8 - d_8h_{16} + d_8x_8^2) \end{aligned}$$

and

$$(3.13) \quad R_1(g_{24}) = g_{24} + d_8^3 - d_8^2x_8 - d_8h_{16} + d_8x_8^2$$

hold. The element x_{36} is shown to be R_1 -invariant and we have

$$(3.14) \quad \begin{aligned} \sigma_{18}^S = & -x_{36} + x_{20}(-x_4^2x_8 - x_8^2) - y_{22}x_4y_{10} - y_{26}y_{10} \\ & - x_4^3x_8^3 + x_4^2x_8y_{10}^2 + x_4x_8^4 - x_8^2y_{10}^2. \end{aligned}$$

For the sake of completeness, we list the result of σ_j^S for $j > 18$, where x_{48} and x_{54} are the elements found in [MS] and will be studied in §4.

$$\begin{aligned} \sigma_{20}^S &= -x_{20}^2 + x_{20}x_4x_8^2 + x_4^2x_8^4 + x_8^5 - x_{20}y_{10}^2 - x_4x_8^2y_{10}^2 - y_{10}^4, \\ \sigma_{21}^S &= x_{20}x_4x_8y_{10} + x_4^3y_{10}^3 + x_4x_8y_{10}^3 - x_{20}y_{22} - x_4x_8^2y_{22} + y_{10}^2y_{22} + x_8^2y_{26}, \\ \sigma_{22}^S &= -x_{20}x_8^3 + x_8^3y_{10}^2, \\ \sigma_{23}^S &= -x_4^2x_8y_{10}^3 - x_8^2y_{10}^3 + x_{20}y_{26} - y_{10}^2y_{26}, \\ \sigma_{24}^S &= x_{48} - x_{20}x_8y_{10}^2 + x_4x_8^3y_{10}^2 - x_4^2y_{10}^4 + x_8y_{10}^4 + y_{22}y_{26}, \\ \sigma_{25}^S &= -x_{20}y_{10}^3 + y_{10}^5, \\ \sigma_{26}^S &= -x_8^4y_{10}^2 + x_4x_8y_{10}^4 + y_{10}^3y_{22} - x_8^2y_{10}y_{26} - y_{26}^2, \\ \sigma_{27}^S &= -x_{54}. \end{aligned}$$

4. THE INVARIANT SUBALGEBRA $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$

Notation Let Ω be a field extension over a field k . For elements x_1, \dots, x_n of Ω , we denote an algebra generated by x_1, \dots, x_n over k by $k[x_1, \dots, x_n]$ and its quotient field by $k(x_1, \dots, x_n)$. In this section, we note that we use the notations $k[x_1, \dots, x_n]$ and $k(x_1, \dots, x_n)$ by the above extended meaning.

For a field k and Ω , we take \mathbb{Z}_3 and $\mathbb{Z}_3(t, t_1, \dots, t_6)$ respectively. We put $K = \mathbb{Z}_3(x_4, x_8, y_{10}, x_{20}, y_{22}, x_{36})$ and $L = \mathbb{Z}_3(t, x_4, x_8, y_{10}, h_{12}, h_{16})$. We see $K \subset L$.

Lemma 4.1. *We have $L = K(t)$. The extension degree $[L : K]$ is at most 27.*

Proof. From the definition, we see $K \subset L$ and $t \in L$. It means $K(t) \subset L$, where $K(t)$ is a field generated by K and t in L . To show $L \subset K(t)$, it is sufficient to show that $h_{12} \in K$. Because it implies that $h_{16} = (x_{20} - h_{12}x_8)/x_4 \in K(t)$.

Replacing h_{18} by its defining expression (2.14) in $y_{22} = h_{12}y_{10} - h_{18}x_4$, we obtain

$$(4.1) \quad y_{22} = (y_{10} + tx_8 + t^3x_4)h_{12} - t(x_{20} - x_4x_8^2) - t^3x_4^2x_8 - t^5x_4x_8 + t^7x_4^2 - t^9x_4.$$

Hence

$$(4.2) \quad (y_{10} + tx_8 + t^3x_4)h_{12} = y_{22} + t(x_{20} - x_4x_8^2) + t^3x_4^2x_8 + t^5x_4x_8 - t^7x_4^2 + t^9x_4.$$

Thus $h_{12} \in K(t)$ and so does h_{16} . We have shown that $L = K(t)$.

Now we shall find a equation of t over K .

$$(3.16)' \quad 0 = x_{36} + g_{24}x_4^3 - h_{12}^3 + h_{12}^2x_4x_8 + h_{12}x_{20}x_4 - h_{12}x_4^2x_8^2.$$

Replacing g_{24} by its defining expression (3.11), $h_{16}x_4$ by $x_{20} - h_{12}x_8$ and $h_{18}x_4$ by $-y_{22} + h_{12}y_{10}$, we obtain a polynomial in $t, x_4, x_8, y_{10}, x_{20}, y_{22}, x_{36}$ and h_{12} . Since its degree with respect to h_{12} is 3, we multiply (3.12)' by $(y_{10} + tx_8 + t^3x_4)^3$. Then by making use of (4.2) we obtain a polynomial in $\mathbb{Z}_3[t, x_4, x_8, y_{10}, x_{20}, y_{22}, x_{36}]$ of degree 27 with respect to t :

$$(4.3) \quad \begin{aligned} 0 = & x_{36}y_{10}^3 + x_{20}y_{22}x_4y_{10}^2 + y_{22}^2x_4x_8y_{10} - y_{22}^3 - y_{22}x_4^2x_8^2y_{10}^2 + tx_4^3x_8^4y_{10}^2 \\ & - tx_4^4x_8y_{10}^4 + tx_{20}^2x_4y_{10}^2 + tx_{20}x_4^2x_8^2y_{10}^2 + tx_{20}y_{22}x_4x_8y_{10} + ty_{22}^2x_4x_8^2 \\ & - ty_{22}x_4^2x_8^3y_{10} - ty_{22}x_4^3y_{10}^3 - t^2x_4^3y_{10}^5 + t^2x_{20}x_4^3y_{10}^3 - t^3x_4^3x_8y_{10}^4 \\ & - t^3x_4^4x_8^3y_{10}^2 + t^3x_4^5y_{10}^4 - t^3x_{20}^2x_4x_8^2 - t^3x_{20}^3 - t^3x_{20}x_4^2x_8^4 + t^3x_{20}x_4^3x_8y_{10}^2 \\ & - t^3x_{20}y_{22}x_4^2y_{10} + t^3x_{36}x_8^3 + t^3y_{22}^2x_4^2x_8 + t^4x_4^3x_8^2y_{10}^3 + t^4x_4^5x_8y_{10}^3 \\ & - t^4x_{20}^2x_4^2y_{10} + t^4x_{20}x_4^2y_{10}^3 + t^4x_{20}y_{22}x_4^2x_8 - t^4y_{22}x_4^2x_8y_{10}^2 - t^5x_4^3x_8^3y_{10}^2 \\ & + t^5x_{20}x_4^3x_8^3 - t^6x_4^4x_8y_{10}^3 - t^6x_4^6y_{10}^3 - t^6x_{20}x_4^2x_8^2y_{10} - t^6x_{20}x_4^4x_8y_{10} \\ & + t^6x_{20}y_{22}x_4^3 + t^6y_{22}x_4^2x_8^3 - t^6y_{22}x_4^3y_{10}^2 + t^6y_{22}x_4^4x_8^2 - t^7x_4^3x_8^5 + t^7x_4^3y_{10}^4 \\ & + t^7x_4^4x_8^2y_{10}^2 - t^7x_4^5x_8^4 + t^7x_{20}^2x_4^3 + t^7x_{20}x_4^3y_{10}^2 - t^7x_{20}x_4^4x_8^2 + t^9x_4^3x_8^2y_{10}^2 \\ & - t^9x_4^4x_8^4 - t^9x_4^5x_8y_{10}^2 + t^9x_4^6x_8^3 + t^9x_{20}x_4^2y_{10}^2 + t^9x_{20}x_4^3x_8^2 \\ & + t^9x_{20}x_4^5x_8 + t^9x_{36}x_4^3 - t^9y_{22}x_4^2x_8y_{10} + t^9y_{22}x_4^4y_{10} - t^{10}x_4^3x_8^3y_{10} \\ & - t^{10}x_4^4y_{10}^3 + t^{10}x_4^5x_8^2y_{10} - t^{10}x_4^7x_8y_{10} + t^{10}x_{20}x_4^2x_8y_{10} - t^{10}x_{20}x_4^4y_{10} \\ & - t^{10}y_{22}x_4^2x_8^2 + t^{10}y_{22}x_4^4x_8 - t^{10}y_{22}x_4^6 - t^{11}x_4^6y_{10}^2 + t^{11}x_{20}x_4^6 + t^{12}x_4^3x_8^3y_{10}^3 \\ & - t^{12}x_4^4x_8^2y_{10} + t^{12}x_4^8y_{10} - t^{12}x_{20}x_4^3y_{10} - t^{12}y_{22}x_4^3x_8 - t^{12}y_{22}x_4^5 \\ & - t^{13}x_4^3x_8y_{10}^2 + t^{13}x_4^5y_{10}^2 + t^{13}x_4^6x_8^2 + t^{13}x_4^8x_8 + t^{13}x_{20}x_4^3x_8 - t^{13}x_{20}x_4^5 \\ & + t^{15}x_4^3x_8^3 - t^{15}x_4^4y_{10}^2 + t^{15}x_4^7x_8 - t^{15}x_4^9 + t^{15}x_{20}x_4^4 + t^{18}x_4^3x_8y_{10} \\ & + t^{18}x_4^5y_{10} + t^{19}x_4^3x_8^2 + t^{19}x_4^5x_8 + t^{21}x_4^4x_8 + t^{21}x_4^6 - t^{27}x_4^3. \end{aligned}$$

Hence we conclude $[L : K] = [K(t) : K] \leq 27$. □

Corollary 4.2. *The elements $x_4, x_8, y_{10}, x_{20}, y_{22}, x_{36}$ are algebraically independent over \mathbb{Z}_3 .*

Proof. Since L is an algebraic extension of K and the transcendental degree $\text{tr deg}_{\mathbb{Z}_3} L$ of L is 6, we have $\text{tr deg}_{\mathbb{Z}_3} K = 6$. Furthermore K is generated by the 6 elements $x_4, x_8, y_{10}, x_{20}, y_{22}, x_{36}$ over \mathbb{Z}_3 . So we get Corollary 4.2. \square

Lemma 4.3. $\left[L : \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)} \right] = 27$.

Proof. We have the inclusions of fields

$$\begin{aligned} \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)} &\subset \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(\text{Spin}(10))} \\ &= \mathbb{Z}_3(t, x_4, x_8, y_{10}, h_{12}, h_{16}) = L \subset \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5) \end{aligned}$$

In order to apply the Galois theory to our case, we need to show that $W(E_6)$ acts on $H^*(BT; \mathbb{Z}_3)$ as automorphisms, where T is the maximal torus of E_6 and the quotient field of $H^*(BT; \mathbb{Z}_3)$ is $\mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)$.

When we define $H = \{g \in W(E_6) \mid gx = x, \forall x \in H^*(BT; \mathbb{Z}_3)\}$, H is a normal subgroup of $W(E_6)$. According to [C], the Weyl group $W(E_6)$ contains the simple group $SU_4(2)$ with index 2. Hence H is $SU_4(2)$ or $\{e\}$. Suppose $H = SU_4(2)$, then $W(E_6)$ acts on $H^*(BT; \mathbb{Z}_3)$ as an automorphism with the order 2. This is a contradiction from the argument of §1. Thus we can apply the Galois theory in our context.

Remark. This fact, however, can be checked by appealing to Lemma 10.7.1 of [S]. Thus we can apply Galois theory in our context (thanks to Proposition 1.2.4 of [S]).

As is well known [F], the order of $W(E_6)$ is $2^7 \cdot 3^4 \cdot 5$ and that of $W(\text{Spin}(10))$ is $2^7 \cdot 3 \cdot 5$. Noting that $L = K(t, t_1, t_2, t_3, t_4, t_5)^{W(\text{Spin}(10))}$, we have

$$\left[L : \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)} \right] = \frac{|W(E_6)|}{|W(\text{Spin}(10))|} = 3^3.$$

\square

Lemma 4.4. $K = \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)}$.

Proof. From Lemma 4.1, we have

$$K \subset \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)} \subset L = K(t).$$

Hence $\left[L : \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)} \right]$ divides $[L : K]$. Further we have

$$[L : K] \leq \left[L : \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)} \right] = 27$$

from Lemmas 4.1 and 4.3. It follows that $K = \mathbb{Z}_3(t, t_1, t_2, t_3, t_4, t_5)^{W(E_6)}$. \square

To state the next lemma, we use the following notation:

Let A be a ring and S be a set. Then denote by $A\{s\}$ a free A -module $\bigoplus_{s \in S} A\{s\}$ with a basis S . we set

$$\begin{aligned} H &= \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}], \\ M &= \mathbb{Z}_3[x_4, x_8, y_{10}, x_{20}, y_{22}, x_{36}], \\ \widetilde{M} &= \mathbb{Z}_3[x_4, x_4^{-1}][x_8, y_{10}, x_{20}, y_{22}, x_{36}]. \end{aligned}$$

Lemma 4.5. (1) $L = K\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$. In particular $t^i h_{12}^j$, $0 \leq i \leq 8$, $0 \leq j \leq 2$ is linearly independent over M and \widetilde{M} .

(2) $H \subset \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$.

Proof. (1) Since $K(t)$ is an algebraic extension of K and $h_{12} \in L = K(t)$, we see

$$L = K(t) = K[t] = K[h_{12}, t].$$

From the relation

$$(4.2)' \quad t^9 x_4 = t^7 x_4^2 - t^5 x_4 x_8 + t^3 h_{12} x_4 - t^3 x_4^2 x_8 + t h_{12} x_8 - t x_{20} + t x_4 x_8^2 + h_{12} y_{10} - y_{22},$$

we obtain $L = K[h_{12}]\{t^i \mid 0 \leq i \leq 8\}$. At the same time, the formula (4.2)' asserts that t is integral over $\widetilde{M}[h_{12}]$.

From (3.11) and (3.12), we have $h_{12}^3 \in K\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$ and hence $L = K[h_{12}]\{t^i \mid 0 \leq i \leq 8\} = K\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$.

On the other hand, this means that $\dim_K L \leq 27$.

From Lemmas 4.3 and 4.4, $\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$ is linear independent over K . Noting that quotient field of M and \widetilde{M} is K , the last statement is clear.

(2) First we show that $\mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}] \in \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$. To prove this assertion, it is enough to show that $t^n, h_{12}^n \in \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$.

As is shown in the above proof, we have $t^n \in M[h_{12}]\{t^i \mid 0 \leq i \leq 8\}$. Next we will show that $\widetilde{M}[h_{12}]\{t^i \mid 0 \leq i \leq 8\} = \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$. We substitute $h_{16} = (x_{20} - h_{12} x_8)/x_4$ for h_{16} in (2.14) and obtain (recalling $h_{18} = t(h_{16} - x_8^2) + t^3(-h_{12} + x_4 x_8) + t^5 x_8 - t^7 x_4 + t^9$)

$$(2.14)' \quad h_{18} = t(x_{20} - h_{12} x_8)/x_4 - t x_8^2 + t^3(-h_{12} + x_4 x_8) + t^5 x_8 - t^7 x_4 + t^9.$$

We replace h_{18} in (3.11) by the righthand side of (2.14)' and obtain (recalling

$$(3.11)' \quad \begin{aligned} g_{24} = & -t^8 x_8 + t^7 y_{10} - t^6 x_4^3 + t^6 x_4 x_8 + t^4 x_4^2 x_8 - t^4 x_8^2 + t^3 h_{18} + t^3 x_4^2 y_{10} \\ & - t^3 x_8 y_{10} + t^2 x_{20} - t^2 y_{10}^2 - t y_{22} - t x_4 x_8 y_{10} \\ (3.11)' \quad g_{24} = & t^{12} - t^{10} x_4 + t^7 x_4 x_8 + t^7 y_{10} - t^7 h_{12} - t^6 x_4^3 + t^6 x_4 x_8 + t^4 x_4^2 x_8 + t^4 x_8^2 \\ & + t^4 x_{20}/x_4 - t^4 h_{12} x_8/x_4 + t^3 x_4^2 y_{10} - t^3 x_8 y_{10} + t^2 x_{20} - t^2 y_{10}^2 - t y_{22} - t x_4 x_8 y_{10}. \end{aligned}$$

We replace g_{24} in (3.12) by the righthand side of (2.15)' and obtain (recalling $x_{36} = -g_{24} x_4^3 + h_{12}^3 - h_{12}^2 x_4 x_8 - h_{12} x_{20} x_4 + h_{12} x_4^2 x_8^2$)

$$(4.4) \quad \begin{aligned} h_{12}^3 = & -t^8 x_4^3 x_8 + t^7 x_4^3 y_{10} - t^6 x_4^6 + t^6 x_4^4 x_8 + t^4 x_4^5 x_8 - t^4 x_4^3 x_8^2 + t^3 h_{12} x_4^2 y_{10} - t^3 y_{22} x_4^2 \\ & + t^3 x_4^5 y_{10} - t^3 x_4^3 x_8 y_{10} + t^2 x_{20} x_4^3 - t^2 x_4^3 y_{10}^2 - t y_{22} x_4^3 - t x_4^4 x_8 y_{10} \\ & + h_{12}^2 x_4 x_8 + h_{12} x_{20} x_4 - h_{12} x_4^2 x_8^2 + x_{36}. \end{aligned}$$

From (4.2)' and (4.4), when we note that $t^n \in \widetilde{M}\{t^i, h_{12} t^i \mid 0 \leq i \leq 8\}$ for $n = 10, 12$, we have $h_{12}^n \in \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$.

$$\widetilde{M}[h_{12}]\{t^i \mid 0 \leq i \leq 8\} = \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}.$$

□

Theorem 4.6. $H^*(BT; \mathbb{Z}_3)^{W(E_6)} = H \cap \widetilde{M}$.

Proof. From Lemma 4.4, we have $H^*(BT; \mathbb{Z}_3)^{W(E_6)} = H \cap K$.

Using Lemma 4.5 (2), we see

$$H \cap K \subset \widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\} \cap K.$$

We note that K is the quotient field of \widetilde{M} and that $\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\}$ is linear independent over \widetilde{M} and K from Lemma 4.5 (1). Hence we get

$$\widetilde{M}\{t^i h_{12}^j \mid 0 \leq i \leq 8, 0 \leq j \leq 2\} \cap K = \widetilde{M}.$$

This implies that $H^*(BT; \mathbb{Z}_3)^{W(E_6)} \subset H \cap \widetilde{M}$.

On the hand, we see that $H \cap \widetilde{M} \subset H \cap K = H^*(BT; \mathbb{Z}_3)^{W(E_6)}$. So we have proved the theorem. \square

We state the main theorem in this section, that is, $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ is generated by the thirteen elements:

$$x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}, x_{36}, x_{48}, x_{54}, y_{58}, y_{60}, y_{64}, y_{76}.$$

These elements except x_4, x_8, y_{10}, x_{36} are defined in $H \cap \widetilde{M}$ as follows:

$$\begin{aligned} x_{20} &= h_{12}x_8 + h_{16}x_4, \\ y_{22} &= h_{12}y_{10} - h_{18}x_4, \\ y_{26} &= (x_{20}y_{10} - y_{22}x_8)/x_4 = h_{16}y_{10} + h_{18}x_8, \\ (4.5) \quad x_{48} &= (-x_{36}x_8^3 + x_{20}^3 + x_{20}^2x_4x_8^2 + x_{20}x_4^2x_8^4)/x_4^3 \\ &= h_{16}^3 + h_{16}^2x_8^2 + h_{16}x_8^4 + g_{24}x_8^3, \end{aligned}$$

$$\begin{aligned} (4.6) \quad x_{54} &= (x_{36}y_{10}^3 - y_{22}^3 + x_{20}y_{22}x_4y_{10}^2 + y_{22}^2x_4x_8y_{10} - y_{22}x_4^2x_8^2y_{10}^2)/x_4^3 \\ &= h_{18}^3 - h_{16}h_{18}y_{10}^2 + h_{18}^2x_8y_{10} + h_{18}x_8^2y_{10}^2 - g_{24}y_{10}^3, \end{aligned}$$

$$\begin{aligned} (4.7) \quad y_{58} &= (x_{36}x_8^2y_{10} - x_{20}^2y_{22})/x_4 \\ &= (h_{16}^2h_{18} - h_{16}^2x_8y_{10} - h_{16}x_8^3y_{10} - g_{24}x_8^2y_{10} + y_{26}x_8^3)x_4^2 \\ &\quad + (h_{16}^2y_{22} + h_{16}h_{18}x_{20} + x_{20}y_{26}x_8 + y_{22}x_8^4)x_4 \\ &\quad + (h_{16}x_{20}y_{22} + h_{18}x_{20}^2 + x_{20}y_{22}x_8^2), \end{aligned}$$

$$\begin{aligned} (4.8) \quad y_{64} &= (y_{58}y_{10} - x_{20}y_{22}y_{26} - y_{22}^2x_8^3)/x_4 \\ &= (x_{36}x_8^2y_{10}^2 + x_{20}^2y_{22}y_{10} + x_{20}y_{22}^2x_8 - y_{22}^2x_4x_8^3)/x_4^2 \\ &= (h_{16}^2h_{18}y_{10} - h_{16}^2x_8y_{10}^2 + h_{16}h_{18}y_{26} - h_{16}x_8^3y_{10}^2 - g_{24}x_8^2y_{10}^2)x_4 \\ &\quad + (y_{26}^2x_8 + y_{26}x_8^3y_{10})x_4 + (h_{16}y_{22}y_{26} + h_{18}x_{20}y_{26} \\ &\quad - y_{22}y_{26}x_8^2 + y_{22}x_8^4y_{10}), \end{aligned}$$

$$\begin{aligned} (4.9) \quad y_{60} &= (x_{36}x_8y_{10}^2 - x_{20}y_{22}^2)/x_4 \\ &= (-h_{16}h_{18}^2 - h_{16}^2y_{10}^2 - h_{16}x_8^2y_{10}^2 - g_{24}x_8y_{10}^2 + y_{26}^2 + x_8^2y_{10}y_{26})x_4^2 \\ &\quad + (h_{16}h_{18}y_{22} + h_{18}^2x_{20} - y_{22}y_{26}x_8 + y_{22}x_8^3y_{10})x_4 \end{aligned}$$

$$\begin{aligned}
& + (-h_{16}y_{22}^2 - h_{18}x_{20}y_{22} + y_{22}^2x_8^2). \\
(4.10) \quad y_{76} &= (y_{58}y_{22} + y_{60}x_{20} + x_{20}y_{22}^2x_8^2)/x_4 \\
&= (x_{36}x_{20}x_8y_{10}^2 + x_{36}y_{22}x_8^2y_{10} + x_{20}^2y_{22}^2 + x_{20}y_{22}^2x_4x_8^2)/x_4^2 \\
&= (-h_{16}^2y_{10} - h_{16}x_8^2y_{10} - g_{24}x_8y_{10} + x_8^2y_{26})y_{26}x_4^2 \\
&+ (h_{16}^2h_{18}y_{22} - h_{16}h_{18}^2x_{20} + h_{16}^2y_{22}x_8y_{10} + h_{16}y_{22}x_8^3y_{10})x_4 \\
&+ (g_{24}y_{22}x_8^2y_{10} + x_{20}y_{26}^2)x_4 + (h_{16}^2y_{22}^2 - h_{16}h_{18}x_{20}y_{22} + h_{18}^2x_{20}^2 - y_{22}^2x_8^4).
\end{aligned}$$

For a while, we denote by A the algebra generated by the thirteen elements

$$\{x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}, x_{36}, x_{48}, x_{54}, x_{58}, y_{60}, y_{64}, y_{76}\}.$$

Our aim is to prove that $A = H \cap \widetilde{M} = H^*(BT; \mathbb{Z}_3)^{W(E_6)}$.

We put $C = \mathbb{Z}_3[x_4, x_8, y_{10}]\{1, x_{20}, x_{20}^2, y_{22}, y_{22}^2, x_{20}y_{22}, y_{58}, y_{60}, y_{76}\} \oplus \mathbb{Z}_3[x_8, y_{10}]\{y_{26}, y_{26}^2, x_{20}y_{26}, y_{22}y_{26}, y_{64}\}$, where it is considered as a formal one. Then we define a \mathbb{Z}_3 -linear map $\sigma : C \otimes \mathbb{Z}_3[x_{36}, x_{48}, x_{54}] \rightarrow A$ by $\sigma(x^I \gamma_i \otimes y^J) = x^I y^J \gamma_i$, where $x^I \in \mathbb{Z}_3[x_4, x_8, y_{10}]$, $y^J \in \mathbb{Z}_3[x_{36}, x_{48}, x_{54}]$ and γ_i is an element of $\{1, x_{20}, x_{20}^2, y_{22}, y_{22}^2, x_{20}y_{22}, y_{58}, y_{60}, y_{76}\}$ and $\{y_{26}, y_{26}^2, x_{20}y_{26}, y_{22}y_{26}, y_{64}\}$.

Proposition 4.7. $\sigma : C \otimes \mathbb{Z}_3[x_{36}, x_{48}, x_{54}] \rightarrow A$ is an isomorphism as a \mathbb{Z}_3 -linear map. Hence its Poincaré polynomial PS is given by

$$PS(A) = \frac{g(t)}{(1-t^4)(1-t^8)(1-t^{10})(1-t^{36})(1-t^{48})(1-t^{54})},$$

where $g(t) = 1 + t^{20} + t^{22} + t^{26} - t^{30} + t^{40} + t^{42} + t^{44} + t^{46} + t^{48} - t^{50} - t^{56} + t^{58} + t^{60} + t^{64} - t^{68} + t^{76}$.

To prove the proposition, we prepare some lemmas

Lemma 4.8. σ is injective.

Proof. We will show that the elements

$$\begin{aligned}
(4.11) \quad & A_1, A_2x_{20}, A_3x_{20}^2, A_4y_{22}, A_5y_{22}^2, A_6x_{20}y_{22}, A_7y_{58}, A_8y_{60}, A_9y_{76}, \\
& B_1y_{26}, B_2y_{26}^2, B_3x_{20}y_{26}, B_4y_{22}y_{26}, B_5y_{64}
\end{aligned}$$

are linearly independent over \mathbb{Z}_3 , where A_i is a monomial of $x_4, x_8, y_{10}, x_{36}, x_{48}, x_{54}$ and B_i a monomial of $x_8, y_{10}, x_{36}, x_{48}, x_{54}$.

To show it, we introduce a filtration to $H = \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]$. We define a weight w by

$$w(t) = w(h_{12}) = w(h_{16}) = 0, \quad w(x_4) = 1, \quad w(x_8) = 2, \quad w(y_{10}) = 3.$$

For a monomial of H , we define the weight by $w(x_4^{i_1}x_8^{i_2}y_{10}^{i_3}t^{i_4}h_{12}^{i_5}h_{16}^{i_6}) = i_1w(x_4) + i_2w(x_8) + i_3w(y_{10})$. For an element $x = \sum \lambda_i x_i \in H$ ($\lambda_i \in \mathbb{Z}_3$), where x_i is a monomial, we define that $w(x) = \inf_i w(x_i)$. Then we introduce in H by setting

$$F^p H = \{x \in H \mid w(x) \geq p\}.$$

Since $F^{p+1}H$ is an ideal of F^pH ($p \geq 0$), the associated graded module $grH = \bigoplus_{p \geq 0} F^pH/F^{p+1}H$ is an algebra. In grH , we have

$$\begin{aligned}
 (4.12) \quad & x_{20} = x_4 h_{16}, \\
 & y_{22} = -x_4 h_{18}, \quad (h_{18} = t^9 - t^3 h_{12} + t h_{16},) \\
 & y_{26} = x_8 h_{18}, \\
 & x_{36} = h_{12}^3, \quad x_{48} = h_{16}^3, \quad x_{54} = h_{18}^3, \\
 & y_{58} = x_4^2 h_{16}^2 h_{18}, \quad y_{64} = x_4 x_8 h_{16} h_{18}^2, \quad y_{60} = x_4^2 h_{16} h_{18}^2, \quad y_{76} = x_4^2 h_{16}^2 h_{18}^2.
 \end{aligned}$$

For simplicity we denote $\mathbb{Z}_3[x_4, x_8, y_{10}, x_{36}, x_{48}, x_{54}]$ by S and $\mathbb{Z}_3[x_8, y_{10}, x_{36}, x_{48}, x_{54}]$ by T . Let

$$\begin{aligned}
 (4.13)' \quad & A_1 + A_2 x_{20} + A_3 x_{20}^2 + A_4 y_{22} + A_5 y_{22}^2 + A_6 x_{20} y_{22} + A_7 y_{58} + A_8 y_{60} + A_9 y_{76} \\
 & + B_1 y_{26} + B_2 y_{26}^2 + B_3 x_{20} y_{26} + B_4 y_{22} y_{26} + B_5 y_{64}
 \end{aligned}$$

equal to zero. Then we see that $A_1 + A'_2 h_{16} + A'_3 h_{16}^2 + A'_4 h_{18} + A'_5 h_{18}^2 + A'_6 h_{16} h_{18} + A'_7 h_{16}^2 h_{18} + A'_8 h_{16} h_{18}^2 + A'_9 h_{16}^2 h_{18}^2 + \tilde{B}_1 h_{18} + \tilde{B}_2 h_{18}^2 + B'_3 h_{16} h_{18} + B'_4 h_{18}^2 + B'_5 h_{16} h_{18}^2 = 0$, where $A_i \in S$, $A'_i = x_4 A_i \in x_4 S$, $A''_i = x_4^2 A_i \in x_4^2 S$ and $B_i, \tilde{B}_1 = x_8 B_1, \tilde{B}_2 = x_8^2 B_2 \in T$, $B'_i = x_4 x_8 B_i \in x_4 T$.

Noting $h_{18} = t^9 - t^3 h_{12} + t h_{16}$ in grH , the elements $x_4, x_8, y_{10}, h_{12}, h_{16}, h_{18}$ are algebraically independent over \mathbb{Z}_3 . So we obtain $A_1 = A'_2 = A'_3 = A'_4 = A'_5 = A'_6 = A'_7 = A'_8 = A'_9 = 0$ at once. With the respect to the coefficients of h_{18} , we note the fact $x_4 S \cap T = \{0\}$ and get $A'_4 = \tilde{B}_1 = 0$. We can prove that the other coefficients are zero in a very similar way. Hence we see that the elements indicated at (4.11) are linearly independent. \square

Next we will prove that σ is surjective. This part is the most crucial step in this section. Before proving it, we observe the following lemma by a tedious calculation.

Lemma 4.9. *There are relations in A :*

$$\begin{aligned}
 x_{54} x_8^3 &= x_{48} y_{10}^3 - y_{26}^3 - y_{26}^2 x_8^2 y_{10} - y_{26} x_8^4 y_{10}^2, \\
 y_{58} x_8 &= -x_{48} x_4^2 y_{10} + x_{20}^2 y_{26} + x_{20} y_{22} x_8^3 + x_{20} y_{26} x_4 x_8^2 + y_{22} x_4 x_8^5 + y_{26} x_4^2 x_8^4, \\
 y_{58} y_{10} &= y_{64} x_4 + x_{20} y_{22} y_{26} + y_{22}^2 x_8^3, \\
 y_{58} y_{22} &= -y_{60} x_{20} + y_{76} x_4 - x_{20} y_{22}^2 x_8^2, \\
 y_{58} y_{26} &= x_{48} y_{22} x_4 y_{10} + y_{64} x_{20} - x_{20} y_{22} y_{26} x_8^2 - y_{22}^2 x_8^5 - y_{22} y_{26} x_4 x_8^4, \\
 y_{58}^2 &= -x_{48} y_{60} x_4^2 - x_{48} x_{20}^2 x_4^2 y_{10}^2 - x_{48} x_{20} y_{22}^2 x_4 - x_{48} x_{20} x_4^3 x_8^2 y_{10}^2 + y_{76} x_{20}^2 \\
 &\quad - x_{20}^3 y_{22} y_{26} x_8 + x_{20}^3 y_{26}^2 x_4 - x_{20}^3 x_4 x_8^4 y_{10}^2 - x_{20}^2 y_{22}^2 x_8^4 + x_{20}^2 y_{22} y_{26} x_4 x_8^3 \\
 &\quad + x_{20}^2 x_4^2 x_8^6 y_{10}^2 - x_{20}^2 y_{26}^2 x_4^2 x_8^2, \\
 y_{58} y_{60} &= -x_{48} x_{54} x_4^4 - x_{48} y_{22}^3 x_4 - x_{48} y_{22}^2 x_4^2 x_8 y_{10} + x_{48} y_{22} x_4^3 x_8^2 y_{10}^2 \\
 &\quad + x_{48} y_{22} y_{26} x_4^3 y_{10} - x_{48} y_{26} x_4^4 x_8 y_{10}^2 + x_{54} x_{20}^3 x_4 + x_{54} x_{20}^2 x_4^2 x_8^2 - y_{76} x_{20} y_{22} \\
 &\quad - x_{20}^2 y_{22} y_{26}^2 x_4 - x_{20} y_{22}^2 y_{26} x_4^3 x_8^3 + x_{20} y_{26}^3 x_4^3 x_8 - y_{22}^3 x_4 x_8^6 + y_{22}^2 x_4^2 x_8^7 y_{10} \\
 &\quad - y_{22}^2 y_{26} x_4^2 x_8^5 - y_{22} y_{26}^2 x_4^3 x_8^6 y_{10} + y_{22} y_{26}^2 x_4^3 x_8^4 + y_{26}^2 x_4^4 x_8^5 y_{10} + y_{26}^3 x_4^4 x_8^3,
 \end{aligned}$$

$$\begin{aligned}
y_{58}y_{64} &= x_{48}x_{54}x_4^3x_8 + x_{48}y_{22}^3x_8 - x_{48}y_{22}^2y_{26}x_4 - x_{48}y_{22}x_4^2x_8^3y_{10}^2 \\
&\quad + x_{48}y_{22}y_{26}x_4^2x_8y_{10} + x_{54}x_{20}^3x_8 + x_{54}x_{20}^2x_4x_8^3 - x_{20}^2y_{22}y_{26}^2x_8 \\
&\quad - x_{20}y_{22}^2y_{26}x_8^4 + y_{22}^3x_8^7 + y_{22}^2x_4x_8^8y_{10} + y_{22}^2y_{26}x_4x_8^6 + y_{22}y_{26}x_4^2x_8^7y_{10}, \\
y_{58}y_{76} &= x_{48}x_{54}x_{20}x_4^3 - x_{48}y_{60}y_{22}x_4 + x_{48}x_{20}y_{22}^3 + x_{48}y_{22}^2x_4^2x_8^3y_{10} \\
&\quad + x_{48}y_{22}y_{26}x_4^3x_8^2y_{10} + x_{54}x_{20}^4 + x_{54}x_{20}^3x_4x_8^2 + x_{54}x_{20}^2x_4^2x_8^4 \\
&\quad + x_{20}^2y_{22}^2y_{26}x_8^3 - x_{20}^2y_{22}y_{26}^2x_4x_8^2 - x_{20}y_{22}^3x_8^6 + x_{20}y_{22}^2y_{26}x_4x_8^5 \\
&\quad - x_{20}y_{22}y_{26}^2x_4^2x_8^4 - y_{22}^3x_4x_8^8 + y_{22}^2y_{26}x_4^2x_8^7 - y_{22}y_{26}^2x_4^3x_8^6, \\
y_{60}x_8 &= y_{64}x_4 - x_{20}y_{22}y_{26} + y_{22}^2x_8^3, \\
y_{60}y_{10} &= x_{54}x_4^2x_8 + y_{22}^2x_8^2y_{10} - y_{22}^2y_{26} + y_{22}y_{26}x_4x_8^3y_{10}^2 - y_{22}x_4x_8y_{10}, \\
y_{60}y_{26} &= -y_{64}y_{22} + x_{54}x_{20}x_4x_8 + y_{22}^2x_8^4y_{10} + y_{22}y_{26}x_4x_8^3y_{10} - y_{22}y_{26}^2x_4x_8, \\
y_{60}^2 &= x_{48}y_{22}^2x_4^2y_{10}^2 - x_{48}y_{22}x_4^3x_8y_{10}^3 + x_{54}y_{58}x_4^2 + x_{54}x_{20}^2y_{22}x_4 - x_{54}x_{20}y_{22}x_4^2x_8^2 \\
&\quad + y_{76}y_{22}^2 + x_{20}y_{22}^3y_{26}x_8 - x_{20}y_{22}^2y_{26}^2x_4 - y_{22}^4x_8^4 - y_{22}^3x_4x_8^5y_{10} + y_{22}^3y_{26}x_4x_8^3 \\
&\quad + y_{22}^2x_4^2x_8^6y_{10}^2 + y_{22}y_{26}x_4^3x_8^5y_{10}^2 + y_{22}y_{26}^2x_4^3x_8^3y_{10} + y_{22}y_{26}^3x_4^3x_8, \\
y_{60}y_{64} &= -x_{48}x_{54}x_4^3y_{10} - x_{48}y_{22}^3y_{10} - x_{48}y_{22}^2y_{10} - x_{48}y_{22}^2x_4x_8y_{10}^2 + x_{48}y_{22}x_4^2x_8^2y_{10}^3 \\
&\quad - x_{48}y_{26}x_4^3x_8y_{10}^3 - x_{54}x_{20}^2y_{22}x_8 + x_{54}x_{20}^2y_{26}x_4 + x_{54}x_{20}y_{26}x_4^2x_8^2 \\
&\quad + y_{22}^3x_8^6y_{10} - y_{22}^3y_{26}x_8^4 + y_{22}^2x_4x_8^7y_{10}^2 - y_{22}^2y_{26}x_4x_8^5y_{10} - y_{22}y_{26}x_4^2x_8^6y_{10}^2 \\
&\quad - y_{22}y_{26}^2x_4^2x_8^4y_{10} - y_{22}y_{26}^3x_4^2x_8^2 + y_{26}^2x_4^3x_8^5y_{10}^2 + y_{26}^3x_4^3x_8^3y_{10} + y_{26}^4x_4^3x_8, \\
y_{60}y_{76} &= -x_{48}x_{54}y_{22}x_4^3 - x_{48}y_{22}^4 - x_{48}y_{22}^3x_4x_8y_{10} + x_{48}y_{22}^2y_{26}x_4^2y_{10} \\
&\quad + x_{48}y_{22}y_{26}x_4^3x_8y_{10}^2 + x_{54}y_{58}x_{20}x_4 - x_{54}x_{20}^3y_{22} - x_{20}^2y_{22}^2y_{26}^2 - x_{20}y_{22}^3y_{26}x_8^3 \\
&\quad + x_{20}y_{22}^2y_{26}^2x_4x_8^2 - x_{20}y_{22}y_{26}^3x_4^2x_8 - y_{22}^4x_8^6 - y_{22}^3x_4x_8^7y_{10} + y_{22}^2y_{26}x_4^2x_8^6y_{10} \\
&\quad - y_{22}y_{26}^2x_4^3x_8^5y_{10} - y_{22}y_{26}^3x_4^3x_8^3, \\
y_{64}x_8 &= -x_{48}x_4y_{10}^2 + x_{20}y_{26}^2 + y_{22}x_8^5y_{10} - y_{22}y_{26}x_8^3 + y_{26}x_4x_8^4y_{10} + y_{26}^2x_4x_8^2, \\
y_{64}y_{10} &= x_{54}x_4x_8^2 + y_{22}x_8^4y_{10}^2 - y_{22}y_{26}x_8^2y_{10} + y_{22}y_{26}^2, \\
y_{64}y_{26} &= x_{48}y_{22}y_{10}^2 + x_{54}x_{20}x_8^2 + y_{22}y_{26}^2x_8^2, \\
y_{64}^2 &= x_{48}^2y_{26}^2x_4^2y_{10}^2 - x_{48}x_{54}x_4^2x_8y_{10} + x_{48}y_{22}^2y_{26}y_{10} + x_{48}y_{22}x_4x_8^3y_{10}^3 \\
&\quad - x_{48}y_{22}y_{26}x_4x_8y_{10}^2 - x_{48}y_{26}x_4^2x_8^2y_{10}^3 + x_{54}x_{20}^2y_{26}x_8 \\
&\quad + x_{20}y_{26}^4x_4 + y_{22}^2x_8^8y_{10}^2 + y_{22}^2y_{26}x_8^6y_{10} - y_{22}^2y_{26}^2x_8^4 - y_{22}y_{26}x_4x_8^7y_{10}^2 \\
&\quad + y_{26}^2x_4^2x_8^6y_{10}^2 - y_{26}^3x_4^2x_8^4y_{10} - y_{26}^4x_4^2x_8^2, \\
y_{64}y_{76} &= x_{48}x_{54}y_{22}x_4^2x_8 - x_{48}x_{54}y_{26}x_4^3 + x_{48}y_{22}^3y_{26} - x_{48}y_{22}^2x_4x_8^3y_{10}^2 \\
&\quad - x_{48}y_{22}y_{26}x_4^2x_8^2y_{10}^2 - x_{48}y_{22}y_{26}^2x_4^2y_{10} - x_{48}y_{26}^2x_4^3x_8y_{10}^2 + x_{54}x_{20}^3y_{26} \\
&\quad + x_{54}x_{20}^2y_{26}x_4x_8^2 + x_{20}^2y_{22}y_{26}^3 - x_{20}y_{22}^2y_{26}^2x_8^3 + x_{20}y_{22}y_{26}^3x_4x_8^2 \\
&\quad + x_{20}y_{26}^4x_4x_8 - y_{22}^3x_8^8y_{10} + y_{22}^3y_{26}x_8^6 - y_{22}^2y_{26}x_4x_8^7y_{10} - y_{22}^2y_{26}^2x_4x_8^5 \\
&\quad + y_{22}y_{26}^2x_4^2x_8^6y_{10} - y_{22}y_{26}^3x_4^2x_8^4 + y_{26}^3x_4^3x_8^5y_{10} + y_{26}^4x_4^3x_8^3, \\
y_{76}x_8 &= -x_{48}y_{22}x_4y_{10} + y_{64}x_{20} + x_{20}y_{22}y_{26}x_8^2 + y_{22}^2x_8^5 + y_{22}y_{26}x_4x_8^4, \\
y_{76}y_{10} &= x_{54}x_{20}x_4x_8 + y_{64}y_{22} + y_{22}^2x_8^4y_{10} + y_{22}^2y_{26}x_8^2 + y_{22}y_{26}x_4x_8^3y_{10} \\
&\quad - y_{22}y_{26}^2x_4x_8,
\end{aligned}$$

$$\begin{aligned}
y_{76}y_{26} &= x_{48}y_{22}^2y_{10} + x_{54}x_{20}^2x_8 - x_{20}y_{22}y_{26}^2x_8 + y_{22}^2y_{26}x_8^4 + y_{22}y_{26}^2x_4x_8^3, \\
y_{76}^2 &= x_{48}x_{54}x_{20}y_{22}x_4^2 + x_{48}y_{60}y_{22}^2 + x_{54}y_{58}x_{20}^2 + x_{54}x_{20}^3y_{22}x_8^2 + x_{20}^2y_{22}^2y_{26}^2x_8^2 \\
&\quad + x_{20}y_{22}^3y_{26}x_8^5 + x_{20}y_{22}^2y_{26}^2x_4x_8^4 + y_{22}^4x_8^8 - y_{22}^3y_{26}x_4x_8^7 + y_{22}^2y_{26}^2x_4^2x_8^6.
\end{aligned}$$

To prove Proposition 4.7, we calculate the Poincaré polynomial of A . From Lemmas 4.8 and 4.9, we have

$$PS(A) = PS(C \otimes \mathbb{Z}_3[x_{36}, x_{48}, x_{54}]) = PS(C) \cdot PS(\mathbb{Z}_3[x_{36}, x_{48}, x_{54}]).$$

Let $A\{s\}$ be a free A -module with the set of generators S . Then we have $PS(A\{s\}) = PS(A)PS(\{s\})$. Using this formula, we have

$$PS(C) = \frac{1 + t^{20} + t^{40} + t^{22} + t^{44} + t^{42} + t^{58} + t^{60} + t^{76}}{(1 - t^4)(1 - t^8)(1 - t^{10})} + \frac{t^{26} + t^{52} + t^{48} + t^{46} + t^{64}}{(1 - t^8)(1 - t^{10})}.$$

We obtain the Poincaré polynomial of A by a direct calculation.

Lemma 4.10. σ is surjective.

Proof. The result of Lemma 4.9 claims that $\text{Im } \sigma$ is an algebra. From the definition, A is the minimal algebra of which contains the generators $x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}, x_{36}, x_{48}, x_{54}, y_{58}, y_{60}, y_{64}, y_{76}$. It means that $A \subset \text{Im } \sigma$. But $\text{Im } \sigma \subset A$ is obvious. So we obtain that σ is surjective. \square

Proposition 4.11. We have the following presentation of A .

$$\begin{aligned}
A = & \left(\mathbb{Z}_3[x_4, x_8, y_{10}]\{1, x_{20}, x_{20}^2, y_{26}, y_{26}^2, x_{20}y_{26}, y_{58}, y_{64}, y_{76}\} \right. \\
& \left. \oplus \mathbb{Z}_3[x_8, y_{10}]\{y_{22}, y_{22}^2, x_{20}y_{22}, y_{22}y_{26}, y_{60}\} \right) \otimes \mathbb{Z}_3[x_{36}, x_{48}, x_{54}].
\end{aligned}$$

Proof. We show that the elements shown at the proposition are linear independent. To do this, we introduce a filtration into $H = \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]$. We define a weight v by $v(t) = 1, v(x_4) = +\infty, v(x_8) = 6, v(y_{10}) = 7, v(h_{12}) = v(h_{16}) = 9$. Then we can define a filtration to H as defined in the proof of Lemma 4.8. It is immediate from the definition that at grH we have

$$\begin{aligned}
(4.13) \quad & x_{20} \equiv x_8h_{12}, \quad y_{22} \equiv y_{10}h_{12}, \quad y_{26} \equiv x_8h_{18}, \\
& y_{58} \equiv x_8^2h_{12}^2h_{18}, \quad y_{60} \equiv x_8y_{10}h_{12}^2h_{18}, \quad y_{64} \equiv x_8^2h_{12}h_{18}^2, \quad y_{76} \equiv x_8x_4^2h_{12}^2h_{18}^2, \\
& x_{36} \equiv h_{12}^3, \quad x_{48} \equiv h_{16}^3, \quad x_{54} \equiv h_{18}^3.
\end{aligned}$$

As is Lemma 4.8 shown, we can show that the monomials in the above presentation are linearly independent.

We calculate the Poincaré polynomial of the presentation. It is given by

$$\begin{aligned}
& \left(\frac{1 + t^{20} + t^{40} + t^{22} + t^{44} + t^{42} + t^{58} + t^{60} + t^{76}}{(1 - t^4)(1 - t^8)(1 - t^{10})} + \frac{t^{26} + t^{52} + t^{48} + t^{46} + t^{64}}{(1 - t^8)(1 - t^{10})} \right) \\
& \quad \times \frac{1}{(1 - t^{36})(1 - t^{48})(1 - t^{54})}.
\end{aligned}$$

We see that it coincides with that of A . Hence we have completed the proof. \square

Lemma 4.12. *Let*

$$\mathbb{Z}_3[x_4, x_{36}, x_{48}, x_{54}] \otimes \{\mathbb{Z}_3[x_8, y_{10}]\{\sigma_i \mid 1 \leq i \leq 9\} \oplus \mathbb{Z}_3[y_{10}]\{\beta_j \mid 1 \leq j \leq 5\}\}$$

be the presentation shown at Proposition 3.11. If we assume that

$$\sum_{i=1}^9 f_i \cdot g_i \sigma_i + \sum_{j=1}^5 f_j \cdot g_j \beta_j \equiv 0 \pmod{x_4 \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]},$$

where $f_i, f_j \in \mathbb{Z}_3[x_4, x_{36}, x_{48}, x_{54}]$, $g_i \in \mathbb{Z}_3[x_8, y_{10}]$, $g_j \in \mathbb{Z}_3[y_{10}]$, then $f_\lambda \equiv 0 \pmod{x_4 \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]}$ holds for $\lambda = i, j$.

Proof of Lemma 4.12. It is sufficient that we prove the statement in $\mathcal{G}rH$. Hence it is enough to prove that

$$\sum_{i=1}^9 g_i(x_8^3, y_{10}^3) \tilde{\sigma}_i + \sum_{j=1}^5 g_j(y_{10}^3) \tilde{\beta}_j \equiv 0 \pmod{x_4 grH}$$

implies $g_i = g_j = 0$.

From (4.13), all the $\tilde{\sigma}_i$ and $\tilde{\beta}_j$ contain no the x_4 -factor. Then we can prove the statement in a similar way to the first part of Proposition 4.11.

Theorem 4.13. (1) $H^*(BT; \mathbb{Z}_3)^{W(E_6)}$ is generated by the thirteen elements

$$x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}, x_{36}, x_{48}, x_{54}, y_{58}, y_{60}, y_{64}, y_{76},$$

where these elements are defined from (4.5) to (4.10).

(2) The relations are given at Lemma 4.9

Proof. From Theorem 4.6, we have

$$H^*(BT; \mathbb{Z}_3)^{W(E_6)} = H \cap \widetilde{M},$$

where $H = \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]$, $\widetilde{M} = \mathbb{Z}_3[x_4, x_4^{-1}][x_8, y_{10}, x_{20}, y_{22}, x_{36}]$. Let A be an algebra generated by x_4, \dots, y_{76} . Then $\mathbb{Z}_3[x_4, x_4^{-1}] \otimes_{\mathbb{Z}_3[x_4]} A$ contains \widetilde{M} . Hence it is enough

to prove that $H \cap \left(\mathbb{Z}_3[x_4, x_4^{-1}] \otimes_{\mathbb{Z}_3[x_4]} A \right) = A$.

In other word, when we use the presentation given at Proposition 4.11, it is sufficient to show that

$$\sum_{i=1}^9 f_i \cdot g_i(x_8^3, y_{10}^3) \sigma_i + \sum_{j=1}^5 f_j \cdot g_j(y_{10}^3) \beta_j \equiv 0 \pmod{x_4 H},$$

$f_i, f_j \in \mathbb{Z}_3[x_4, x_{36}, x_{48}, x_{54}]$ implies $f_\lambda \equiv 0 \pmod{x_4 H}$ for all $\lambda = i, j$.

It is just the same as the statement of Lemma 4.12. Hence we have completed the proof. \square

5. APPENDIX

From now on, coefficients will be in \mathbb{Z}_3 throughout the calculation.

Denote by c_i and p_i the elementary symmetric functions on $\{t_i\}$ and $\{t_j^2\}$, respectively. Then we have

$$(5.1) \quad \begin{aligned} c_2 &= p_1 - c_1^2, \\ c_4 &= -p_2 + c_1^4 + c_1^2 p_1 + c_1 c_3 + p_1^2, \\ c_3 c_5 &= p_4 - c_4^2, \\ c_3^2 &= p_3 + c_1^6 + c_1^3 c_3 - c_1^2 p_2 - c_1 c_3 p_1 + c_1 c_5 - p_1^3 + p_1 p_2. \end{aligned}$$

The first six $W(E_6)$ -invariant elements $x_4, x_8, y_{10}, x_{20}, y_{22}$ and y_{26} are easily found as in Section 1:

$$(5.2) \quad \begin{aligned} x_4 &= p_1, \\ x_8 &= p_2 - p_1^2, \\ y_{10} &= c_5 - x_8 t - x_4 t^3, \\ x_{20} &= h_{12} x_8 + h_{16} x_4, \\ y_{22} &= h_{12} y_{10} - h_{18} x_4, \\ y_{26} &= h_{16} y_{10} + h_{18} x_8, \end{aligned}$$

where

$$(5.3) \quad \begin{aligned} h_{12} &= p_3 + y_{10} t - x_8 t^2 - x_4 t^4, \\ h_{16} &= p_4 + y_{10} t^3 - x_8 t^4 - x_4 t^6, \\ h_{18} &= t(h_{16} - x_8^2) + t^3(-h_{12} + x_4 x_8) + t^5 x_8 - t^7 x_4 + t^9. \end{aligned}$$

Observe that there holds the following relation:

$$(5.4) \quad -x_{20} y_{10} + y_{22} x_8 + y_{26} x_4 = 0.$$

Calculation of σ_j^S is carried out as follows.

Put $w_1 = t - c_1$ and $w_i = t + c_1 - t_{i-1}$ for $2 \leq i \leq 6$. The set

$$S = \{w_i + w_j, c_1 - w_i, -c_1 - w_i ; i < j\}$$

is invariant as a set under the action of $W(E_6)$ (see [TW]). Therefore the elementary symmetric functions σ_j^S on S are invariant under the action of $W(E_6)$.

Put

$$(5.5) \quad \begin{aligned} P &= 1 + \sum_{j=1}^{27} \sigma_j^S = \prod_{y \in S} (1 + y) \\ &= \prod_{1 \leq i < j \leq 6} (1 + w_i + w_j) \prod_{1 \leq j \leq 6} (1 + c_1 - w_j) \prod_{1 \leq j \leq 6} (1 - c_1 - w_j). \end{aligned}$$

(We shall rewrite P as far as possible in terms of the six invariant elements in (5.2).)

P is expressed by t and c_i 's and has 2600 terms. We replace c_2 , c_4 , c_5 , c_3^2 and c_3y_{10} in P by the following relations:

$$(5.6) \quad \begin{aligned} c_2 &= -c_1^2 + x_4, \\ c_4 &= c_1^4 + c_1c_3 + c_1^2x_4 - x_8, \\ c_5 &= y_{10} + x_8t + x_4t^3, \end{aligned}$$

$$(5.7) \quad \begin{aligned} c_3^2 &= h_{12} + t^4x_4 + t^2x_8 - ty_{10} + x_4x_8 + t^3x_4c_1 + tx_8c_1 - x_4^2c_1^2 \\ &\quad - x_4c_1c_3 - x_8c_1^2 + y_{10}c_1 + c_1^6 + c_1^3c_3, \end{aligned}$$

$$(5.8) \quad \begin{aligned} c_3y_{10} &= h_{16} + t^6x_4 + t^4x_8 - t^3y_{10} - x_8^2 - t^4x_4c_1^2 - t^3x_4c_1^3 \\ &\quad - t^3x_4c_3 - t^2x_8c_1^2 - tx_8c_1^3 - tx_8c_3 + ty_{10}c_1^2 - h_{12}c_1^2 + x_4x_8c_1^2 \\ &\quad + x_4c_1^6 - x_4c_1^3c_3 - x_8c_1c_3 - y_{10}c_1^3 + c_1^8 \end{aligned}$$

and we see that all c_3 's are cancelled.

We use two more relations to eliminate c_1 's.

Replacing c_3^2 in the equality $c_3^2y_{10} - c_3 \cdot c_3y_{10} = 0$, we get a relation:

$$(5.9) \quad \begin{aligned} h_{16}c_3 &= t^9x_4 + t^7x_4^2 + t^7x_8 - t^6y_{10} - t^5x_4x_8 + t^3h_{12}x_4 + t^3h_{16} + t^3x_4^2x_8 + th_{12}x_8 \\ &\quad + tx_4x_8^2 - ty_{10}^2 + h_{12}y_{10} + x_4x_8y_{10} + t^7x_4c_1^2 + t^6x_4c_1^3 + t^6x_4c_3 + t^5x_4c_1^4 + t^5x_8c_1^2 \\ &\quad - t^4x_4^2c_1^3 - t^4x_4x_8c_1 - t^4x_4c_1^5 - t^4x_4c_1^2c_3 + t^4x_8c_1^3 + t^4x_8c_3 - t^4y_{10}c_1^2 - t^3h_{12}c_1^2 \\ &\quad - t^3x_4^2c_1^4 - t^3x_4^2c_1^4 + t^3x_4x_8c_1^2 - t^3x_4c_1^3c_3 + t^3x_8c_1^4 - t^3x_8c_1c_3 + t^3c_1^8 - t^2x_4x_8c_1^3 \\ &\quad - t^2x_8^2c_1 - t^2x_8c_1^5 - t^2x_8c_1^2c_3 - t^2y_{10}c_1^4 + th_{12}c_1^4 - th_{16}c_1^2 - tx_4^2x_8c_1^2 + tx_4x_8c_1^4 \\ &\quad + tx_4y_{10}c_1^3 - tx_4c_1^8 + tx_4c_1^5c_3 + tx_8^2c_1^2 + tx_8y_{10}c_1 - tx_8c_1^6 + tx_8c_1^3c_3 - tc_1^{10} \\ &\quad - h_{12}x_4c_1^3 + h_{12}x_8c_1 + h_{12}c_1^5 + h_{12}c_1^2c_3 - h_{16}x_4c_1 - h_{16}c_1^3 - x_4^3c_1^5 - x_4^2x_8c_1^3 \\ &\quad - x_4^2y_{10}c_1^2 - x_4^2c_1^7 - x_4x_8^2c_1 + x_4x_8c_1^5 - x_4x_8c_1^2c_3 - x_4y_{10}c_1^4 - x_4c_1^9 + x_4c_1^6c_3 \\ &\quad + x_8^2c_3 + x_8c_1^7 - x_8c_1^4c_3 + y_{10}^2c_1 - y_{10}c_1^6 - c_1^{11} - c_1^8c_3. \end{aligned}$$

Making use of rewriting c_3^2 , c_3y_{10} and c_3h_{16} , the equality $c_3^2y_{10}^2 - (c_3y_{10})^2 = 0$ gives rise to the following relation.

$$(5.10) \quad \begin{aligned} c_1^{16} &= -t^{12}x_4^2 + t^{10}x_4^3 + t^{10}x_4x_8 - t^9x_4y_{10} - t^8x_8^2 + t^7x_4^2y_{10} - t^7x_8y_{10} + t^6h_{12}x_4^2 \\ &\quad + t^6h_{16}x_4 + t^6x_4^3x_8 - t^6x_4x_8^2 - t^6y_{10}^2 - t^5x_4x_8y_{10} - t^4h_{12}x_4x_8 + t^4h_{16}x_8 \\ &\quad - t^4x_4^2x_8^2 - t^4x_4y_{10}^2 - t^3h_{12}x_4y_{10} - t^3h_{16}y_{10} - t^3x_4^2x_8y_{10} - t^3x_8^2y_{10} + t^2h_{12}x_8^2 \\ &\quad + t^2x_4x_8^3 - t^2x_8y_{10}^2 - th_{12}x_8y_{10} - tx_4x_8^2y_{10} - ty_{10}^3 + h_{12}y_{10}^2 - h_{16}^2 - h_{16}x_8^2 \\ &\quad + x_4x_8y_{10}^2 - x_8^4 - t^{10}x_4^2c_1^2 - t^8x_4^2c_1^4 + t^8x_4x_8c_1^2 - t^7x_4y_{10}c_1^2 - t^6h_{12}x_4c_1^2 \\ &\quad - t^6x_4^4c_1^2 - t^6x_4^3c_1^4 + t^6x_4^2x_8c_1^2 + t^6x_4^2c_1^6 - t^6x_4x_8c_1^4 + t^6x_4c_1^8 - t^6x_8^2c_1^2 \end{aligned}$$

$$\begin{aligned}
& -t^5x_4y_{10}c_1^4 - t^5x_8y_{10}c_1^2 + t^4h_{12}x_4c_1^4 - t^4h_{12}x_8c_1^2 - t^4h_{16}x_4c_1^2 + t^4x_4^3x_8c_1^2 \\
& - t^4x_4^3c_1^6 + t^4x_4^2x_8c_1^4 + t^4x_4^2c_1^8 + t^4x_4x_8^2c_1^2 - t^4x_4x_8c_1^6 - t^4x_4c_1^{10} + t^4x_8c_1^8 \\
& - t^4y_{10}^2c_1^2 - t^3h_{12}x_4x_8c_1 + t^3h_{12}y_{10}c_1^2 - t^3h_{16}x_4^2c_1 + t^3x_4^4c_1^5 - t^3x_4^3x_8c_1^3 \\
& + t^3x_4^3y_{10}c_1^2 + t^3x_4^3c_1^7 + t^3x_4^2x_8c_1^5 + t^3x_4^2y_{10}c_1^4 - t^3x_4x_8^2c_1^3 - t^3x_4x_8y_{10}c_1^2 \\
& + t^3x_4x_8c_1^7 + t^3x_4y_{10}^2c_1 - t^3x_4y_{10}c_1^6 + t^3x_8y_{10}c_1^4 - t^3y_{10}c_1^8 + t^2h_{12}x_8c_1^4 \\
& - t^2h_{16}x_8c_1^2 - t^2x_4^2x_8^2c_1^2 - t^2x_4^2x_8c_1^6 - t^2x_4x_8^2c_1^4 + t^2x_4x_8c_1^8 + t^2x_8^2c_1^6 - t^2x_8c_1^{10} \\
& - t^2y_{10}^2c_1^4 - th_{12}x_8^2c_1 - th_{12}y_{10}c_1^4 - th_{16}x_4x_8c_1 + th_{16}y_{10}c_1^2 + tx_4^3x_8c_1^5 \\
& - tx_4^2x_8^2c_1^3 + tx_4^2x_8y_{10}c_1^2 + tx_4^2x_8c_1^7 + tx_4^2y_{10}c_1^6 + tx_4x_8^2c_1^5 + tx_4x_8y_{10}c_1^4 \\
& - tx_4y_{10}c_1^8 - tx_8^3c_1^3 + tx_8^2c_1^7 + tx_8y_{10}^2c_1 - tx_8y_{10}c_1^6 + ty_{10}c_1^{10} - h_{12}^2c_1^4 - h_{12}h_{16}c_1^2 \\
& - h_{12}x_4^2c_1^6 - h_{12}x_4x_8c_1^4 + h_{12}x_4c_1^8 - h_{12}x_8^2c_1^2 - h_{12}x_8y_{10}c_1 - h_{12}x_8c_1^6 - h_{12}c_1^{10} \\
& - h_{16}x_4^2c_1^4 - h_{16}x_4y_{10}c_1 - h_{16}x_4c_1^6 + h_{16}x_8c_1^4 + h_{16}c_1^8 - x_4^4c_1^8 + x_4^3x_8c_1^6 + x_4^3y_{10}c_1^5 \\
& - x_4^3c_1^{10} - x_4^2x_8y_{10}c_1^3 - x_4^2y_{10}^2c_1^2 + x_4^2y_{10}c_1^7 + x_4x_8^3c_1^2 + x_4x_8y_{10}c_1^5 - x_4c_1^{14} + x_8^3c_1^4 \\
& - x_8^2y_{10}c_1^3 + x_8y_{10}^2c_1^2 + x_8y_{10}c_1^7 + x_8c_1^{12} + y_{10}^3c_1 - y_{10}^2c_1^6.
\end{aligned}$$

Observe that there is no c_3 in (5.10).

Making use of (5.10), we find that c_1 is cancelled out and $P \in \mathbb{Z}_3[t, x_4, x_8, y_{10}, h_{12}, h_{16}]$. The highest degree of t is 27.

From (5.3) and (5.4) the following rewriting rules are of use.

$$(5.11) \quad t^9 = h_{18} + t^7x_4 - t^5x_8 + t^3h_{12} - t^3x_4x_8 - th_{16} + tx_8^2,$$

$$(5.12) \quad h_{18}x_8 = -h_{16}y_{10} + y_{26},$$

$$h_{18}x_4 = h_{12}y_{10} - y_{22},$$

$$h_{16}x_4 = -h_{12}x_8 + x_{20},$$

$$y_{26}x_4 = x_{20}y_{10} - y_{22}x_8,$$

and, in case of need

$$h_{18}x_{20} = h_{12}y_{26} - h_{16}y_{22}.$$

Now we decompose P by degree and see that σ_j^S for $j = 18, 24, 27$ are not expressed by $x_4, x_8, y_{10}, x_{20}, y_{22}$ and y_{26} .

The element σ_{18}^S is now as follows.

$$\begin{aligned}
\sigma_{18}^S = & -x_{20}x_4^2x_8 - x_{20}x_8^2 - y_{22}x_4y_{10} - y_{26}y_{10} - x_4^3x_8^3 + x_4^2x_8y_{10}^2 + x_4x_8^4 - x_8^2y_{10}^2 \\
& - h_{12}^3 + h_{12}^2x_4x_8 + h_{12}x_{20}x_4 - h_{12}x_4^2x_8^2 - t^8x_4^3x_8 + t^7x_4^3y_{10} - t^6x_4^6 + t^6x_4^4x_8 \\
& + t^4x_4^5x_8 - t^4x_4^3x_8^2 + t^3h_{12}x_4^2y_{10} - t^3y_{22}x_4^2 + t^3x_4^5y_{10} - t^3x_4^3x_8y_{10} \\
& + t^2x_{20}x_4^3 - t^2x_4^3y_{10}^2 - ty_{22}x_4^3 - tx_4^4x_8y_{10}.
\end{aligned}$$

Collect the terms with t in σ_{18}^S and we put

$$\begin{aligned}
(5.13) \quad g_{24} = & -t^8x_8 + t^7y_{10} - t^6x_4^3 + t^6x_4x_8 + t^4x_4^2x_8 - t^4x_8^2 + t^3h_{18} + t^3x_4^2y_{10} - t^3x_8y_{10} \\
& + t^2x_{20} - t^2y_{10}^2 - ty_{22} - tx_4x_8y_{10}
\end{aligned}$$

then define

$$\begin{aligned}
 (5.14) \quad x_{36} &= -g_{24}x_4^3 + h_{12}^3 - h_{12}^2x_4x_8 - h_{12}x_{20}x_4 + h_{12}x_4^2x_8^2, \\
 x_{48} &= g_{24}x_8^3 + h_{16}^3 + h_{16}^2x_8^2 + h_{16}x_8^4, \\
 x_{54} &= -g_{24}y_{10}^3 + h_{18}^3 - h_{16}h_{18}y_{10}^2 + h_{18}^2x_8y_{10} + h_{18}x_8^2y_{10}^2.
 \end{aligned}$$

The elementary symmetric functions on S are calculated as follows:

$$\begin{aligned}
 (5.15) \quad \sigma_i^S &= 0 \quad \text{for } 1 \leq i \leq 5 \quad \text{and } i = 7, 10, 11, 13, 19, \\
 \sigma_6^S &= -x_4^3 - x_4x_8, \\
 \sigma_8^S &= -x_4^2x_8 - x_8^2, \\
 \sigma_9^S &= -x_4^2y_{10} - x_8y_{10}, \\
 \sigma_{12}^S &= -x_{20}x_4 + x_4^6 - x_4^4x_8 + x_4y_{10}^2 - x_8^3, \\
 \sigma_{14}^S &= x_{20}(x_4^2 - x_8) - x_4^5x_8 - x_4^3x_8^2 - x_4^2y_{10}^2 + x_8y_{10}^2, \\
 \sigma_{15}^S &= x_{20}y_{10} + y_{22}(x_4^2 + x_8) - x_4^5y_{10} + x_4x_8^2y_{10} - y_{10}^3, \\
 \sigma_{16}^S &= -x_{20}x_4^3 + x_4^3y_{10}^2, \\
 \sigma_{17}^S &= x_{20}x_4y_{10} + y_{22}(x_4^3 - x_4x_8) - y_{26}x_8 \\
 &\quad + x_4^4x_8y_{10} - x_4^2x_8^2y_{10} + x_4y_{10}^3 + x_8^3y_{10}, \\
 \sigma_{18}^S &= -x_{36} - x_{20}x_4^2x_8 - x_{20}x_8^2 - x_4^3x_8^3 \\
 &\quad + x_4x_8^4 + x_4^2x_8y_{10}^2 - x_8^2y_{10}^2 - x_4y_{10}y_{22} - y_{10}y_{26}, \\
 \sigma_{20}^S &= -x_{20}^2 + x_{20}x_4x_8^2 + x_4^2x_8^4 + x_8^5 - x_{20}y_{10}^2 - x_4x_8^2y_{10}^2 - y_{10}^4, \\
 \sigma_{21}^S &= x_{20}x_4x_8y_{10} + x_4^3y_{10}^3 + x_4x_8y_{10}^3 - x_{20}y_{22} - x_4x_8^2y_{22} + y_{10}^2y_{22} + x_8^2y_{26}, \\
 \sigma_{22}^S &= -x_{20}x_8^3 + x_8^3y_{10}^2, \\
 \sigma_{23}^S &= -x_4^2x_8y_{10}^3 - x_8^2y_{10}^3 + x_{20}y_{26} - y_{10}^2y_{26}, \\
 \sigma_{24}^S &= x_{48} - x_{20}x_8y_{10}^2 + x_4x_8^3y_{10}^2 - x_4^2y_{10}^4 + x_8y_{10}^4 + y_{22}y_{26}, \\
 \sigma_{25}^S &= -x_{20}y_{10}^3 + y_{10}^5, \\
 \sigma_{26}^S &= -x_8^4y_{10}^2 + x_4x_8y_{10}^4 + y_{10}^3y_{22} - x_8^2y_{10}y_{26} - y_{26}^2, \\
 \sigma_{27}^S &= -x_{54}.
 \end{aligned}$$

Note that the elements $x_4, x_8, y_{10}, x_{20}, y_{22}, y_{26}, x_{36}, x_{48}$ and x_{54} are also found in Section 3 by making use of Galois theory.

REFERENCES

- [AR] S. Araki, *On the non-commutativity of Pontrjagin rings mod 3 of some compact exceptional groups*, Nagoya Math. J., **17** (1960), 225–260.
- [A] E. Artin, *Galois Theory*, Notre Dame Math. Lectures **2**, 1979.
- [BH] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, Amer. J. Math., **80** (1958), 964–1029.
- [B] N. Bourbaki, *Groupes et algèbres de Lie*, IV–VI, 1968.

- [C] C. Chevalley, *Sur certains groupes simples*, Tohoku Math. J. (2) **7** (1955), 14-66.
- [F] H. Freudenthal and H. de Vries, *Linear Lie groups*, Pure and Applied Math., **35** Academic Press, 1969, New York,.
- [KM] A. Kono - M. Mimura, *Cohomology mod 3 of the classifying space of the Lie group E_6* , Math. Scand., **46**(1980), 223–235.
- [MS] M. Mimura - Y. Sambe, *On the cohomology mod p of the classifying spaces of the exceptional Lie groups, I*, J. Math. of Kyoto Univ., **19** (1979), 553–581.
- [MST1] M. Mimura, Y. Sambe and M. Tezuka: *Cohomology mod 3 of the classifying spaces of the exceptional Lie group of type E_6 , I*, preprint (1986/1991).
- [MST2] M. Mimura, Y. Sambe and M. Tezuka: *Some remarks on the mod 3 cohomology of the classifying space of the exceptional Lie group E_6* , *Proceedings of Workshop in Pure Mathematics, Part III*, **17**(1998), 139 - 159.
- [MST3] M. Mimura, Y. Sambe and M. Tezuka: *Cohomology mod 3 of the classifying spaces of the exceptional Lie group E_6 , I: structure of Cotor*, arXiv:1112.5811.
- [N] M. Nagata: *Theory of commutative fields*. Translated from the 1985 Japanese edition by the author. Translations of Mathematical Monographs, 125. American Mathematical Society, Providence, RI, 1993.
- [RS1] M. Rothenberg - N. E. Steenrod, *The cohomology of the classifying spaces of H -spaces*, Bull. AMS, **71** (1965), 872–875.
- [RS2] M. Rothenberg - N. E. Steenrod, *The cohomology of the classifying spaces of H -spaces*, (mimeographed notes).
- [S] Larry Smith, *Polynomial invariants of finite groups*. Research Notes in Mathematics, **6**. A K Peters, Ltd., Wellesley, MA, 1995.
- [T] H. Toda, *Cohomology of the classifying space of exceptional Lie groups*, Manifolds - Tokyo 1973, 265–271.
- [TW] H. Toda and T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T* , J. Math. of Kyoto Univ., **14** (1974), 257–286.
- [V] H. O. Singh Varma, *The topology of E_{III} and a conjecture of Atiyah and Hirzebruch*, Nederl. Akad. Wet. Indag. Math., **30** (1968), 67–71.

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